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UNIVERSITY OF ALBERTA

IMAGE RESTORATION IN COMPUTED TOMOGRAPHY

BY

SATYAPAL RATHEE



A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Ph.D.

DEPARTMENT OF ELECTRICAL ENGINEERING

EDMONTON, ALBERTA

FALL 1992.



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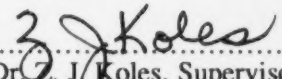
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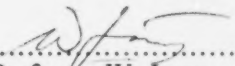
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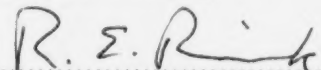
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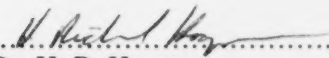
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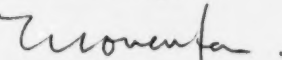
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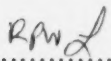

.....
Dr. J. Koles, Supervisor


.....
Professor W. Joerg


.....
Dr. R. E. Rink


.....
Dr. H. R. Hooper


.....
Dr. T. R. Overton


.....
pp Dr. R. E. Kearney

22 May 1992
.....
May 22, 1992

Dedicated to my wife, Pankaj, and my son, Sachin

ABSTRACT

A critical analysis of the spatial resolution achievable in X-ray computed tomography (CT) is presented. In particular, the limit on spatial resolution due to finite focal spot size and detector aperture is reviewed, and an image restoration approach to improve the spatial resolution is described and implemented.

The image restoration problem in CT can be formulated either directly using projection data or indirectly with images reconstructed using filtered backprojection. In order to compare these two formulations, three simple image restoration methods are applied to simulated objects where the point spread function is assumed to be spatially invariant. These three methods are: (1) a constrained least-squares solution using the projection data, (2) a Wiener filter applied to the reconstructed images and (3) a constrained, weighted least-squares solution using the reconstructed images. The third method resulted in the lowest normalized squared error between the original and restored images. In a binary classification scheme, receiver-operator analyses of the restored images obtained from the three methods were found to be statistically equivalent.

Constrained, weighted least-squares solution using experimental CT images requires a measurement of the spatially variant point spread function. For this purpose, two-dimensional X-ray beam profiles at each point in the beam were modeled as convolutions of appropriately scaled focal spot intensity and detector sensitivity distributions. This model was validated by experimental measurement of focal spot intensity distribution. Using the known X-ray beam profiles, projections of a point-object located at various positions in the scan field were computed; the image of the point spread function was then reconstructed from these projections.

In order to restore experimental CT images, weighted least-squares solutions using reconstructed images were obtained in both Cartesian and polar coordinates for a spatially variant point spread function. The restored images of a resolution phantom and of a human cadaver femur, using a storage-based approach in the Cartesian coordinate system, provides better spatial resolution than restoration in polar coordinates.

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LIST OF ABBREVIATIONS

ART	Algebraic Reconstruction Technique
CT	Computed Tomography
CGM	Conjugate Gradient Method
CLS	Constrained Least Squares
DFT	Discrete Fourier Transform
ERF	Edge Response Function
FWHM	Full Width at Half Maximum
kV	kilo Volts
LSF	Line-spread Function
mA	milli Amperes
MAP	Maximum <i>a Posteriori</i>
ME	Maximum Entropy
MTF	Modulation Transfer Function
mSv	milli Sieverts
NPS	Noise Power Spectrum
NSE	Normalized Squared Error
PDF	Probability Distribution Function
POCS	Projection onto Convex Sets
PSF	Point-spread Function
PET	Positron-emission Tomography

ROC	Receiver-operator-characteristics
SCR	Source to center-of-rotation
SEE	Standard Error of Estimate
SPECT	Single Photon-emission Computed Tomography

1. INTRODUCTION

X-ray computed tomography (CT) is concerned with the reconstruction of a section of an object from the projections of X-ray attenuation taken at different orientations. The projection of a two-dimensional (2-D) object is a set of parallel line integrals.

The discovery of X-rays in 1895 by Röntgen, and the theoretical studies of Radon in 1917 [1] concerning the solution of gravitation equations by 'inversion formulae' provided all the basic knowledge for the development of CT. In an independent restatement of the inversion theorem of Radon, Bracewell [2] developed methods for the reconstruction of extraterrestrial microwave sources from signals received on earth by antenna arrays which proved useful for the subsequent application to reconstructive tomography.

A milestone in X-ray CT development was the work of Cormack [3, 4], which was concerned with both theoretical and experimental studies of ionizing radiation-transmission projections, and the reconstruction of a transaxial image of an object from these projections. At that time, however, Cormack was not able to explore his ideas in a practical way since the computational resources were not available. Godfrey Hounsfield began his CT research in the early 1960's, and his work resulted in the first commercial CT scanner in 1971, the EMI CT [5]. Hounsfield and Cormack shared the 1973 Nobel Prize in medicine and physiology for their individual, pioneering contributions to this revolutionary medical diagnostic imaging development.

CT technology has evolved rapidly since 1973, and fast, precise, and accurate measurements of linear attenuation coefficients in a 2-D cross-section of human body are now routine. Currently, the most important application of CT is for high contrast visualization of anatomical structures for diagnostic purposes. Applications of CT scanners also include: radiation therapy planning [6]; blood flow measurements [7]; functional surgery [8]; bone mineral analysis [9]. In radiation therapy for example, quantitative CT images of a part of the body containing a tumor are used to "customize" isodose curves for field size definition, and for subsequent follow-up of the efficacy of treatment procedures. CT scanners with very short scan times (<5 seconds) also provide dynamic measures of the passage of an iodinated contrast medium through an organ for blood flow measurements. In surgery, geometrical information about small anatomical structures obtained from the images is useful.

There are several research applications of CT which require a spatial resolution not

yet achievable with state-of-the-art commercial CT scanners. One such application is the investigation of the structure and density of trabecular bone in diseases such as osteoporosis. A CT scanner provides highly reproducible values for bone density; bone structure, however, cannot be investigated because of inadequate spatial resolution. Several methods for morphological and textural analysis have been proposed for estimating the structural parameters of trabecular bone from high resolution CT images, and a significantly improved separation of healthy subjects from patients has been reported when structural parameters are considered along with conventional bone density measurements [10].

The filtered backprojection method of image reconstruction from projections is most commonly used in CT. This method, in theory, yields an optimal reconstruction in a least-squares sense, as well as being the least computationally intensive. In this method, the projections are first high-pass filtered and then backprojected to reconstruct the image. Ideally, the magnitude of the Fourier transform of the high-pass filter increases linearly with the spatial frequency. In practice, the infinite impulse response, calculated by taking the inverse Fourier transform of the ideal filter, is truncated in the spatial domain by using a suitable window function. Therefore, the spatial resolution depends upon this window function. In the backprojection procedure, an interpolation of the filtered projections is required since the radial position of the pixels in a 2-D image does not coincide with the sample-points of the projections. The spatial resolution in the reconstructed image depends on the method of interpolation used in the backprojection procedure. There are other factors which determine the spatial resolution of CT images including: sampling rate of the projections, the number of projections and the width of X-ray beam profile. However, the ultimate limit to achievable spatial resolution is set by the finite width of the X-ray beam profile. The size of the X-ray beam profile depends, in turn, on the dimensions of the focal spot of the X-ray tube and the detector aperture.

The number of projections collected by the system as well as the number of samples in a projection are important. If a projection is not adequately sampled, then the Fourier spectra of both the projections and the reconstructed image are aliased. Visually, fine detail present in the object is blurred in the reconstructed image when the projections are coarsely sampled. The number of projections taken with a particular CT scanner determines the physical size of the image over which a given spatial resolution can be uniformly achieved. If the number of projections is insufficient 'streak' artifacts occur, particularly for high contrast structures. State-of-the-art commercial CT scanners use advanced technology for

X-ray generation, and for the detector system and scanning geometries. Consequently, a sufficient number of adequately sampled projections is always available with commercial CT systems.

A large X-ray focal spot and a wide detector collimator aperture result in a wide X-ray beam. A wide X-ray beam 'blurs' the ideal projection of line integrals. A finite size X-ray beam low-pass filters the continuous projection of a 2-D object before the data is recorded, thus acting as an anti-aliasing filter. However, when each projection recorded by a CT scanner is adequately sampled, the finite size of the X-ray beam becomes the limiting factor for the achievable physical spatial resolution. Since state-of-the-art CT scanners are capable of collecting a large number of adequately sampled projections, spatial resolution in CT images is inherently limited by the size of the X-ray tube focal spot and the detector collimator design. Spatial resolution can be improved by reducing the sizes of the focal spot and detector collimator; a smaller focal spot results in lower photon flux, and a smaller detector aperture leaves a larger fraction of the photon beam undetected. This results in an increase in both image noise and radiation dose delivered to the patient. For example, an improvement in spatial resolution by a factor of 2 can be obtained by increasing either the statistical noise by a factor of 2.8 or the radiation dose by a factor of 8 [11].

An image restoration based approach to improved spatial resolution (beyond the limit determined by the finite size of the X-ray beam) is presented in this thesis. Classical image restoration techniques are reviewed, and the theory of the filtered backprojection method and a detailed description of the factors affecting the spatial resolution, are presented (Chapter 2). Formulation of the image restoration problem in CT is presented in chapter 3. Simple image restoration methods applied directly to the measured projection data and to the reconstructed images are discussed. In order to determine a suitable formulation and to assess the improvement in the spatial resolution, the restored images for the simulated Shepp and Logan phantom [12] and a random test pattern are compared with the images reconstructed using the filtered backprojection method. The performance of the various image restoration techniques is evaluated using the normalized squared error (NSE) and receiver-operator-characteristics (ROC).

The point spread function (PSF) of a CT scanner is required for the purpose of image restoration. The procedures used to measure the PSF of an actual CT scanner are described in chapter 4. X-ray beam profiles are modeled as the convolution of the measured focal spot intensity and detector sensitivity distributions. The PSF at each pixel in the reconstructed image was then estimated by calculating the projections of a thin, dense wire

using the known beam profiles. The estimated PSF of a real CT scanner was found to be spatially variant.

Due to the spatially variant nature of the PSF, spatially variant image restoration techniques are required to restore the real images. Spatially variant image restoration techniques, in polar and in Cartesian coordinates, are described in chapter 5. These techniques are based on the storage of the pre-computed spatially variant PSF. An image restoration technique which uses a spatially invariant PSF approximation is also given.

Formulation of the 3-D image restoration problem in CT and the methods of 3-D restoration in Cartesian and cylindrical coordinates are discussed in chapter 6. A brief summary of the important results and conclusions are given in chapter 7. The analytical procedures for determining the noise variance and the noise power spectrum (NPS) in the reconstructed images are provided as Appendices I and II respectively. Appendix III presents a brief review of the conjugate gradient algorithm which was used to solve the systems of equations in the various restoration techniques described.

2. BACKGROUND:

Image Reconstruction from Projections, Spatial Resolution and Image Restoration

In this chapter, several different methods of reconstructing functions from projections are summarized. The theory and digital implementation of the most commonly used filtered backprojection method is reviewed. The PSF of X-ray CT is derived as a function of the geometrical parameters, the sampling distance, and the filter and interpolating functions used in the reconstruction method. The effect of these parameters on the theoretical spatial resolution in X-ray CT images is separately evaluated using the PSF. This analysis suggests that the inherent limit to achievable spatial resolution is provided by the sizes of the X-ray tube focal spot and detector collimator aperture. Image restoration can be used to improve the spatial resolution which is limited due to the focal spot and detector aperture. Therefore, a brief overview of image restoration techniques is provided at the end of the chapter.

2.1 IMAGE RECONSTRUCTION FROM PROJECTIONS

The mathematical theory of image reconstruction has applications in several different areas such as medical imaging, electron microscopy [15] and radio astronomy [16]. In diagnostic radiology, X-ray CT [13] is used for visualizing anatomical structures whereas in nuclear medicine, maps of physiological processes are provided by positron-emission tomography (PET) and single photon-emission tomography (SPECT) [14]. In each case, the projections of the distribution of the relevant physical quantity, such as the linear attenuation coefficient, infused radioactive substance (SPECT), surface temperature or the molecular biological structure in a 2-D plane are available. The problem then reduces to the reconstruction of these distributions from the measured projections.

The reconstruction algorithms currently developed may be divided into four groups:

- (1) iterative methods based on an algebraic formulation [17, 18, 19];
- (2) reconstruction in the spatial-frequency domain [20,21];
- (3) the Hankel transform approach [22] and
- (4) the filtered backprojection method in the spatial domain [23, 12].

The iterative methods are used primarily when the projection data is incomplete. The projection data can be incomplete in three different ways: insufficient angular range [24], sparse angular sampling [25] and the hollow projections [26]. The principle behind all these techniques is that the projection data is inadequate to provide a unique distribution of the desired physical quantity. Therefore, *a priori* knowledge, deterministic and/or statistical, is utilized to force the uniqueness of the distribution. For example, the well known algebraic reconstruction technique (ART) is a minimum-norm solution of the algebraic equations relating the projection data to the 2-D distribution. Both the multiplicative ART (MART) [27] and maximum entropy (ME) [28] are similar in that a solution which maximizes the entropy of the image, subject to the linear constraints of the projection data, is determined. There are a number of studies reported for reconstructing the image from sparse and noisy projection data based on maximizing the entropy of the image [29, 30, 31]. Other deterministic *a priori* constraints such as the boundary of object [32], non-negativity and bandlimitedness [33] are also utilized to reduce the artifacts in the image. The performance of ART can be optimized for the application by using the various relaxation parameters [34]. The main advantage of these methods is that the images can be reconstructed to give useful information even when the measured projection data is incomplete and noisy. In this situation, these methods clearly outperform the analytical methods. The disadvantage of these methods is that the computation time is significantly larger than the other analytical techniques. Also due to the iterative nature of the algorithms, the characteristics of noise in the reconstructed images cannot be described analytically.

The spatial-frequency domain algorithm is based upon the Fourier slice theorem [35, 36]. According to this theorem, the Fourier transform of a 2-D function along a slice in the 2-D frequency space is the same as the 1-D Fourier transform of the projection of the function taken at the same angle as the slice. In theory, therefore, the 2-D Fourier transform of a function can be completely specified if the projections are available from all the angles. An inverse 2-D Fourier transform will then provide the distribution of the physical quantity in the spatial coordinates. The major problem with this approach is that only a finite number of projections is available to fill the frequency space in polar coordinates. The readily available 2-D inverse Fourier transform algorithms work on the Cartesian coordinates of the frequency space. An interpolation of the polar frequency components to Cartesian coordinates fails to produce accurate results due to the decreasing number of samples per unit area with increasing distance from the origin. Therefore, due to artifacts resulting from inaccurate interpolation, this method has found limited application. However, some work regarding the accurate interpolation in the frequency space has been reported [21].

The reconstruction from projections can be made in the polar coordinate system using the Hankel transform approach. A 2-D function along a circle in Cartesian coordinates is periodic with a period of 2π . Therefore, at constant radial positions, this function can be expanded as a Fourier series along the angular coordinates. The n^{th} coefficient of this Fourier series, at a fixed radial position, for a 2-D function is related to the corresponding coefficients, at all the radial positions, for the 2-D Fourier transform of the function via the n^{th} order Hankel transform. This forms the basis of reconstructing the image from projections. The 2-D polar form of the Fourier transform of the function is given by the 1-D Fourier transform of the projections. This 2-D Fourier transform is resolved into a Fourier series along the angular coordinate. The resulting coefficients are used to compute the coefficients of the corresponding spatial coordinate Fourier series using the Hankel transform. The spatial domain image is reconstructed from the final coefficients. The digital implementation of this algorithm is shown to be at least two times faster than filtered backprojection method with comparable quality of reconstruction. The main problem with this approach is that there are a number of parameters in the algorithm which need to be simultaneously optimized. In addition, the performance of the algorithm under measurement noise has not yet been studied. Also, this algorithm produces images on a polar grid, thus an interpolation on to a Cartesian grid is needed for display purposes.

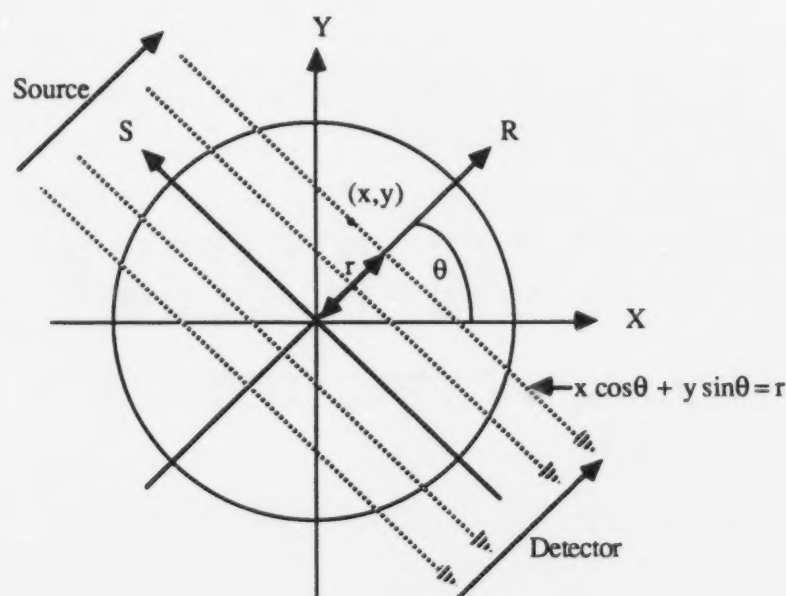


Figure 2.1.1: The parallel beam geometry for measuring projection

The filtered backprojection algorithm for a parallel beam geometry is described in the following discussion, and the technique for re-ordering the fan-beam into parallel beam projections is also provided.

2.1.1 Filtered Backprojection Method

In a parallel beam CT scanner, an X-ray source and a detector, aligned with each other on either side of the object, translate linearly to collect a projection at a particular angle (Figure 2.1.1). Assume that the X-ray source emits I_0 number of photons which travel in straight paths through the object in (x,y) plane. Let $I(r,\theta)$ be the number of photons in the beam after passing through the object along a line given by

$$x \cos\theta + y \sin\theta = r \quad (2.1.1)$$

The projection $p(r,\theta)$ for the linear attenuation coefficient distribution function $f(x,y)$, at an angle θ , is given by

$$p(r,\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \delta(x \cos\theta + y \sin\theta - r) dx dy \quad (2.1.2)$$

where the function δ is unity along the line described by equation (2.1.1) and zero elsewhere. The integral of the function $f(x,y)$ along this line is also known as a ray-sum.

In the R-S coordinate system, since $dx dy = dr ds$ and integration with respect to r vanishes due to the δ function, equation (2.1.2) reduces to the following

$$p(r,\theta) = \int_{-\infty}^{\infty} f(x,y) ds \quad (2.1.3)$$

This projection can be calculated from the incident and transmitted photon intensities, I_0 and $I(r,\theta)$, as follows [13]

$$p(r,\theta) = \ln \{I_0\} - \ln \{I(r,\theta)\} \quad (2.1.4)$$

where $\ln$ denotes the natural logarithm.

Let $F(u,v)$ be the 2-D Fourier transform of the function $f(x,y)$ given by the following equation.

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \exp[-i(ux+vy)] dx dy \quad (2.1.5)$$

where (u,v) are the Cartesian spatial frequency variables defined as radians per unit distance and $i = \sqrt{-1}$.

If the frequency plane is changed to the polar coordinates (ω, θ) such that $u = \omega \cos \theta$ and $v = \omega \sin \theta$ then, using equation (2.1.1), equation (2.1.5) can be written as

$$F(\omega \cos \theta, \omega \sin \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \exp[-i\omega r] ds dr \quad (2.1.6)$$

By interchanging the order of integration and using equation (2.1.3), this can be modified to the following

$$F(\omega \cos \theta, \omega \sin \theta) = \int_{-\infty}^{\infty} p(r, \theta) \exp[-i\omega r] dr \quad (2.1.7)$$

The right hand side of equation (2.1.7) can be recognized as 1-D Fourier transform, $P(\omega, \theta)$, of the projection at angle θ . This is known as the Fourier slice theorem which states that the slice through the 2-D Fourier transform of a 2-D function at angle θ is given by the 1-D Fourier transform of the projection of the function at the same angle.

The function $f(x,y)$ can be computed from its Fourier transform $F(u,v)$ in the following equation

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) \exp[i(ux+vy)] du dv \quad (2.1.8)$$

This equation can be written in the polar form (ω, θ) for the spatial frequency variables [23,36]

$$f(x,y) = \frac{1}{4\pi^2} \int_0^{\pi} \int_{-\infty}^{\infty} F(\omega \cos \theta, \omega \sin \theta) \exp[i(\omega x \cos \theta + \omega y \sin \theta)] |\omega| d\omega d\theta \quad (2.1.9)$$

which can be modified using the Fourier slice theorem and equation (2.1.1) to the following

$$f(x,y) = \frac{1}{2\pi} \int_0^\pi q(r,\theta) d\theta \quad (2.1.10)$$

$$q(r,\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega| P(\omega,\theta) \exp[i\omega r] d\omega \quad (2.1.11)$$

Equation (2.1.10) and (2.1.11) define the filtered backprojection algorithm. The 1-D Fourier transform for each projection, at an angle θ , is first multiplied by a function $|\omega|$ and the result is inverse Fourier transformed to provide the filtered projection $q(r,\theta)$. These filtered projections are then backprojected according to equations (2.1.1) and (2.1.10) to give the linear attenuation coefficient distribution $f(x,y)$.

In practice, a CT scanner is able to collect a finite number of sampled projections of an object at various angles. Assume that a CT scanner collects M projections with the angular spacing between the consecutive projections being $\Delta\theta$. Each projection is sampled at the regular points spaced distance 'a' apart providing a set of N samples. The integrals in equations (2.1.10) and (2.1.11) are thus approximated by finite summations. Digital implementation of the filtered backprojection method produces a different reconstruction depending upon whether the filter function $|\omega|$ is designed in the Fourier domain or in the spatial domain.

The digital implementation of equation (2.1.11) can be directly carried out in the Fourier domain. Equation (2.1.11), in the spatial domain, is a convolution of the filter function and sampled projection. In order to avoid circular convolution, let each sampled projection $p(na, m\Delta\theta)$ be padded with zeroes such that the resulting projections are of size $K \geq 2N-1$. Let $P(k\Delta\omega, m\Delta\theta)$ be the K -point discrete Fourier transform (DFT) of the extended m^{th} projection. Then

$$P(k\Delta\omega, m\Delta\theta) = a \sum_{n=0}^{N-1} p(na, m\Delta\theta) e^{-i2\pi nk/K} \text{ for } k = -\frac{K}{2}, -\frac{K}{2} + 1, \dots, \frac{K}{2} - 1 \quad (2.1.12)$$

where $\Delta\omega$ is the sample spacing in the spatial frequency domain given as $(2\pi/aK)$. Let $H_f(k\Delta\omega)$ be the filter in Fourier domain obtained by sampling the function $|\omega|$ at the same intervals as in equation (2.1.12). The Fourier domain implementation of equation (2.1.11) can then be written as the inverse DFT of the product of $H_f(k\Delta\omega)$ and $P(k\Delta\omega, m\Delta\theta)$.

$$q(na, m\Delta\theta) = \frac{1}{aK} \sum_{k=-K/2}^{K/2-1} H_f(k\Delta\omega) P(k\Delta\omega, m\Delta\theta) e^{i2\pi nk/K} \quad (2.1.13)$$

where $q(na, m\Delta\theta)$ is the filtered projection calculated at the same N positions as the original projection [36].

A more common approach is to implement equation (2.1.11) in the spatial variable r . An impulse response, $h_s(r)$, of the filter function $|\omega|$ is calculated as follows

$$h_s(r) = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} |\omega| e^{i\omega r} d\omega = \frac{B \sin 2\pi Br}{\pi r} - \frac{\sin^2 \pi Br}{\pi^2 r^2} \quad (2.1.14)$$

where the impulse response is bandlimited to B cycles per unit distance.

Since each projection is N samples long, a $(2N-1)$ point sampled impulse response is needed for carrying out the filtering. Assuming that the projections are sampled at the Nyquist rate, it follows that $a=1/(2B)$. Therefore, the sampled impulse response $h_s(na)$ is non-zero only for $n=0$ and the odd values of n .

$$h_s(0) = B^2 \quad (2.1.15a)$$

$$h_s(na) = -\frac{4B^2}{n^2\pi^2} \quad (2.1.15b)$$

where n is always odd in equation (2.1.15b)

Equation (2.1.11) can now be written as a discrete convolution

$$q(na, m\Delta\theta) = \frac{a}{2\pi} \sum_{k=-(N-1)}^{N-1} h_s(ka) p((n-k)a, m\Delta\theta) \quad (2.1.16)$$

The implementation of equation (2.1.16) is preferred since equation (2.1.13) produces an error in mean value and shading in the reconstructed images [38]. The filter used in equation (2.1.13) is a sampled version of the filter function $|\omega|$. The impulse response corresponding to the function $|\omega|$, given in (2.1.14), has an infinite spatial extent. Thus, just as aliasing takes place when a non-bandlimited signal is sampled in the spatial domain causing aliasing in the frequency domain, aliasing will occur in the spatial domain because of sampling in the frequency domain. The aliased impulse response, $h_f(na)$, can be computed from H_f by taking inverse DFT which results in

$$h_f(na) = h_s(na) + h_a(na) \quad (2.1.17)$$

where the aliased portion, $h_a(na)$, of the impulse response is given by

$$h_a(na) = - \sum_{k \neq 0} \frac{B^2 \sin^2 (n-kK)\pi/2}{(n-kK)^2 \pi^2/4} \quad (2.1.18)$$

It can be shown that a few of the low frequency components for $h_a(na)$ are non-zero negative numbers [38]. As a result of this spatial aliasing in the Fourier domain implementation of equation (2.1.11), the reconstructed images depict a negative error in mean value and an upward shading which are distinctly visible in objects comprising large regions of uniform distribution. Therefore, the spatial domain implementation of equation (2.1.11) as in (2.1.16) is preferred to the Fourier domain implementation. The discrete convolution of equation (2.1.16) can be computed using fast Fourier transform (FFT).

After obtaining the filtered projections $q(na, m\Delta\theta)$, the integral in equation (2.1.10) is directly approximated as a finite length summation given by

$$f(ja, ka) = \frac{\Delta\theta}{2\pi} \sum_{m=0}^{M-1} q(ja \cos(m\Delta\theta) + ka \sin(m\Delta\theta), m\Delta\theta) \quad (2.1.19)$$

where $r = ja \cos(m\Delta\theta) + ka \sin(m\Delta\theta)$ is due to equation (2.1.1). The discrete backprojection of equation (2.1.19) requires an interpolation of the filtered projection q since $r = ja \cos(m\Delta\theta) + ka \sin(m\Delta\theta)$ may not coincide exactly with the sampled locations of q at regular intervals a . The nearest neighbor and linear interpolation techniques have been more commonly used since the higher order interpolation methods significantly increase the computational requirements.

The discussion of the filtered backprojection method, so far, was carried out assuming that the parallel projections for the object are available. Modern commercial CT scanners as well as special purpose CT scanners seldom provide the parallel projection data directly. In a CT scanner, there are a number of detectors arranged in an arc D at a fixed distance from the X-ray source S (Figure 2.1.2). Although, an ultra fast CT scanner utilizes a switchable ring of X-ray source and detectors, an X-ray source located at an angle β from the Y-axis is similar to Figure 2.1.2. Therefore, a projection at angular position β of the source is made up of the integrals of the object along the lines which originate at point S and diverge as a function of distance from the source. The collected projection data is denoted in terms of the detector angle with respect to source to center-of-rotation line,

oS , and the angular position of the source with respect to Y-axis. Clearly, the discussion of the filtered backprojection algorithm for fan-beam projection data is different from the discussion in the previous section. However, there is a well defined geometrical relationship between the fan-beam and parallel projections. For example, the fan-beam ray-sum at coordinates (α, β) is the same as the parallel ray-sum at angle θ and a distance r from the origin o where

$$r = oS \sin \alpha \quad (2.1.20a)$$

$$\theta = \beta + \alpha \quad (2.1.20b)$$

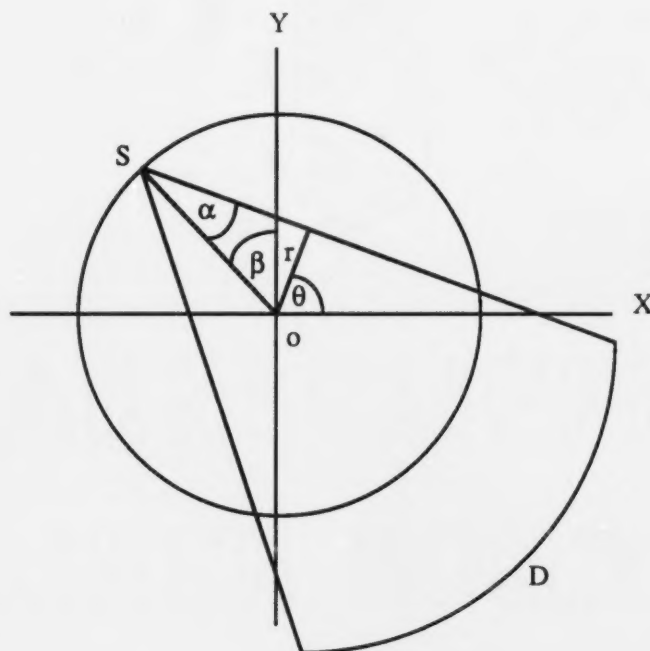


Figure 2.1.2: Relationship between a fan-beam and a parallel projection.

Therefore, the fan-beam projection data can be rebinned into parallel [39, 40] projections and a filtered backprojection method, described in previous section, is then used for image reconstruction. A filtered backprojection algorithm also exists for reconstructing images directly from the fan-beam projections which is faster than the rebinning approach [37]. The rebinning approach is more commonly applied to the multiple rotation CT scanners [39, 40] since complete fan-beam projections are not available until

after the last rotation, and the speed advantage of the fan-beam algorithm is lost.

2.2 SPATIAL RESOLUTION

The spatial resolution of a CT scanner is defined as the smallest distance between two high contrast objects which are distinguishable in a CT image. There are a number of factors which affect the spatial resolution achievable by a CT system. A perfect reconstruction of an object from its projections is only possible when an infinite number of continuous projections is available. A practical CT system collects only a finite number of projections of an object, each sampled with a finite spacing. In addition, the assumption that each ray-sum is the line integral of a 2-D object, is violated in practice because of the finite size of the X-ray beam. The finite size of the X-ray beam results from the finite-size focal spot and a finite detector collimator aperture. Image reconstruction, due to the discrete projection data, must utilize the truncated impulse response of the filter and an interpolation before backprojection. Image reconstruction of the object is carried out only at discrete spatial positions. Each of these factors introduces distortion such that an image only approximately resembles the original object. The most common type of distortion is that the fine details in the object are not visible in the image [42]. Other degradations such as streaking artifacts can occur due to aliasing resulting from both the coarse sampling of each projection and a fewer number of projections [41, 43, 45].

The PSF of a CT system is defined as an image of a point-object. A wider PSF indicates coarser spatial resolution and vice versa. The PSF of a CT system can be derived in terms of focal spot intensity and detector sensitivity distributions, sample spacing of projections, filter function, and interpolation technique. In order to derive an analytical expression for the PSF, an infinite number of projections is assumed. The effect of finite number of projections is mentioned at the end.

Let $N(r)$ and $W(r)$ be the focal spot intensity and detector sensitivity distributions, respectively. In the following discussion, the focal spot intensity distribution is assumed to be a Gaussian function of rms width Δ , and the detector sensitivity distribution as a rectangular function of width r_0 such that

$$N(r) = \frac{1}{\sqrt{2\pi}\Delta} \exp\left(-\frac{r^2}{2\Delta^2}\right) \quad (2.2.1)$$

and

$$W(r) = 1/r_0 \quad \text{for } |r| < r_0/2 \quad (2.2.2)$$

For simplicity, small angle photon scattering will be ignored in the following discussion. Since a beam profile is the integrated intensity of all photons emitted in the focal spot distribution which are acquired within the distribution of the detector sensitivity, the beam profile can then be defined as a convolution of N and W . The measured projections of an object are thus given by the convolution of N , W and $p(r, \theta)$, where $p(r, \theta)$ was defined as the projections of the object made up of line integrals. Let $P(\omega, \theta)$ be the Fourier transform of $p(r, \theta)$, then the Fourier transform of the measured projections $P_s(\omega, \theta)$ is given by the product of the Fourier transforms of N , W and $p(r, \theta)$ since the convolution in the spatial domain is the same as multiplication in frequency domain.

$$P_s(\omega, \theta) = \frac{\sin(\omega r_0/2)}{\omega r_0/2} \exp(-\frac{\omega^2 \Delta^2}{2}) P(\omega, \theta) \quad (2.2.3)$$

Since each projection is sampled at regular distances a , the Fourier transform of the sampled projections, $P_{\text{samp}}(\omega, \theta)$ is given by

$$P_{\text{samp}}(\omega, \theta) = \sum_{n=-\infty}^{\infty} P_s(\omega - \frac{2\pi n}{a}, \theta) \quad (2.2.4)$$

It should be noted that the function $P_{\text{samp}}(\omega, \theta)$ is periodic and continuous over all spatial frequencies.

Similarly, the Fourier transform of the filtered projection can be calculated by multiplying $P_{\text{samp}}(\omega, \theta)$ by the Fourier transform, $H_s(\omega)$, of the filter function $h_s(na)$. An interpolation before backprojection, which is used to convert the sampled, filtered projections into the continuous, filtered projections, can also be considered to be a convolution [42] between the filtered projection and interpolating function, $g(r)$. Therefore, the reconstructed image, $f(r, \phi)$, in polar coordinates is given by

$$f(r, \phi) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} d\theta \int_0^{\infty} e^{j\omega r \cos(\theta-\phi)} G(\omega) H_s(\omega) P_{\text{samp}}(\omega, \theta) d\omega \quad (2.2.5)$$

where $G(\omega)$ is the Fourier transform of function $g(r)$.

Let us assume that a point-object is located at the origin of (x, y) coordinate resulting into projection $p(r, \theta)$ as a δ function. Therefore, the Fourier transform of the measured projection, in equation (2.2.3), does not contain $P(\omega, \theta)$ since $P(\omega, \theta)=1$, resulting in P_{samp}

being independent of θ . Since the purpose of this derivation is to study the behavior of PSF in terms of the parameters r_0 , Δ and a , the PSF can be expressed in terms of the radial distance parameter r by putting $\phi=0$ in equation (2.2.5) as follows

$$f(r,0) = \frac{1}{2\pi} \int_0^{\infty} G(\omega) H_s(\omega) P_{\text{samp}}(\omega) J_0(\omega r) d\omega \quad (2.2.6)$$

where P_{samp} is now expressed as a function of ω only, and

$$J_0(\omega r) = \frac{1}{\pi} \int_0^{\pi} e^{j\omega r \cos \theta} d\theta \quad (2.2.7)$$

is the zeroth-order bessel function [42].

The behavior of the PSF in terms of the focal spot, detector aperture and sampling distance of the projections can be considered by computing equation (2.2.6) for various values of r_0 , Δ and a . In the following discussion, a linear interpolation function and the Fourier transform of the filter function of equation (2.1.15) are used for computing PSF in all cases except when the effects of these functions on the PSF are explicitly analyzed.

2.2.1 Geometric Parameters and Sampling of Projections

The effect on the PSF of varying the width of the detector aperture while holding the focal spot size constant are shown in Figures 2.2.1a and 2.2.1b for sampling distances of 1 mm and 0.2 mm, respectively. The width of the PSF is reduced as the detector aperture is made narrower, at both the sampling distances.

The spatial frequency bandwidth of the projections is determined by the sizes of the focal spot and detector aperture because a wider beam profile produces more blurring in the measured projections. However, the independent spatial frequency bandwidth is limited to half the sampling frequency ($2\pi/a$). If the projections are sampled too coarsely such that half the sampling frequency is smaller than the highest frequency set by the beam profile, aliasing in projection data occurs which causes the PSF to broaden in size and distort in shape. As the sampling distance is decreased, the PSF width becomes smaller because high frequency information in the projections is better preserved. This is especially true when the widths of the focal spot and detector are small as shown in Figure 2.2.2a as the sampling distance is reduced from 2 mm to 1mm. However, as the half the sampling frequency approaches the highest frequency of projection

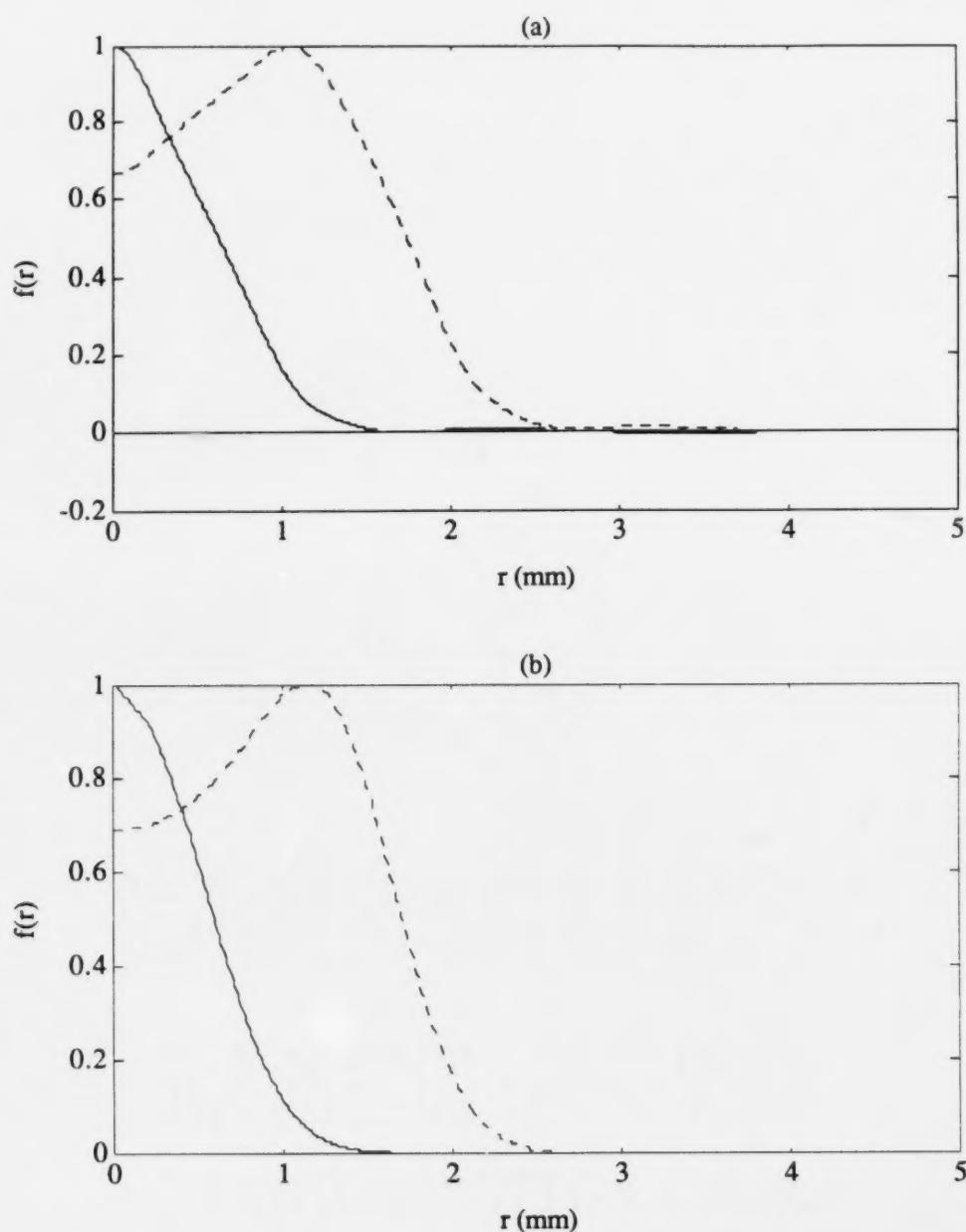


Figure 2.2.1: (a) The PSFs of $a=1$ mm, $\Delta=0.5$ mm, and $r_0=1$ mm (solid), 3 mm (dashed). (b) The PSFs of $a=0.2$ mm, $\Delta=0.5$ mm, and $r_0=1$ mm (solid), 3 mm (dashed). a = sampling distance, Δ = rms width of the focal spot intensity distribution, and r_0 = width of the detector sensitivity distribution.

data, the effect of reducing the sampling distance on the PSF becomes minimal. Therefore, the improvement in the PSF is less significant as the sampling distance is reduced from

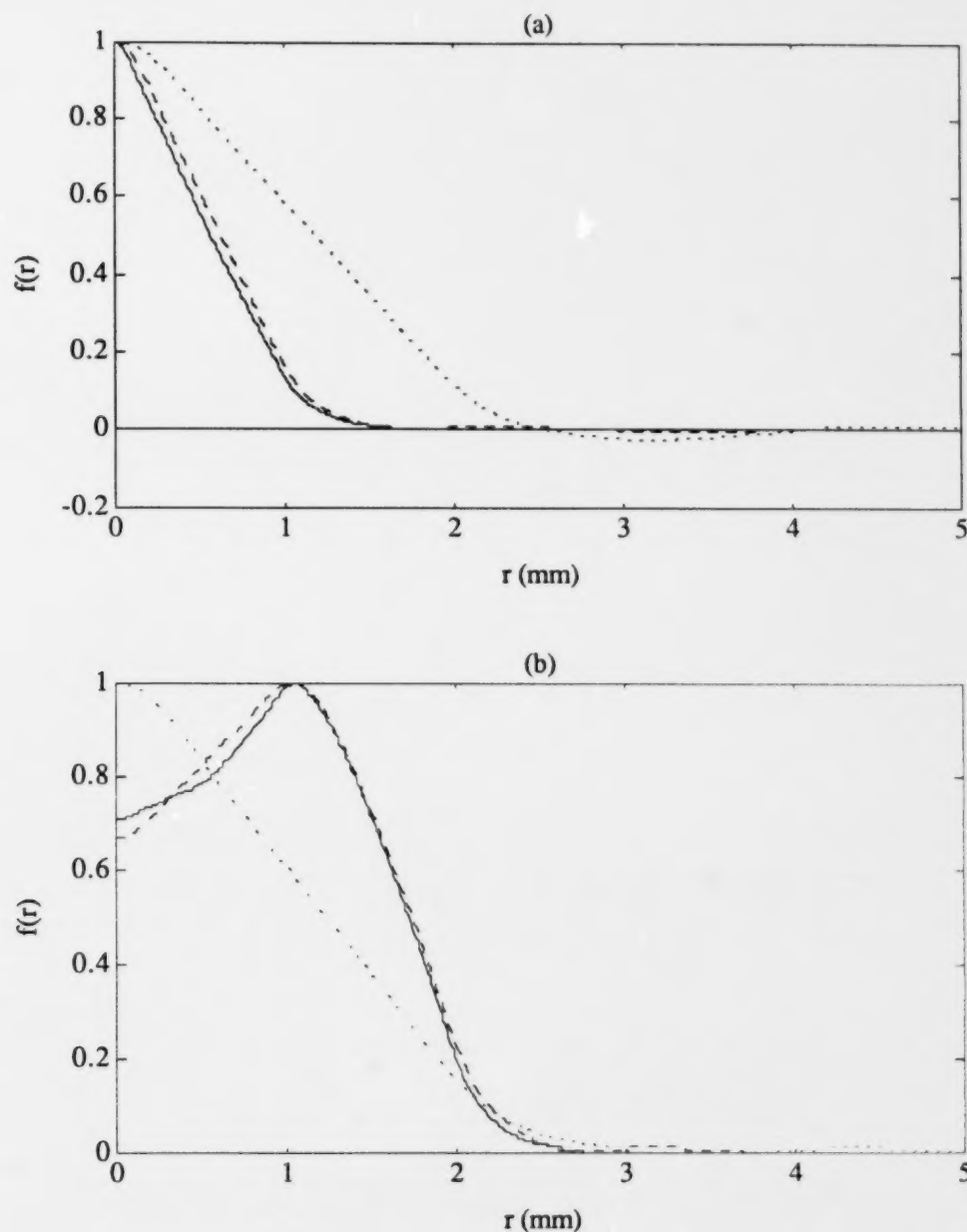


Figure 2.2.2: (a) The PSFs of $r_0=1$ mm, $\Delta=0.5$ mm, and $a=0.5$ mm (solid), 1 mm (dashed), 2 mm (dotted). (b) The PSFs of $r_0=3$ mm, $\Delta=0.5$ mm, and $a=0.5$ mm (solid), 1 mm (dashed), 2 mm (dotted). a = sampling distance, Δ = rms width of the focal spot intensity distribution, and r_0 = width of the detector sensitivity distribution.

1 mm to 0.5 mm (Figure 2.2.2a). For a 3 mm detector aperture width, the highest frequency set by the beam profile becomes smaller compared to the 1 mm case. Therefore,

the baseline width of the PSF in Figure 2.2.2b is almost the same for three different sampling distances. As the sampling distance is made smaller, however, a new form of distortion appears in the PSF as a central depression. This can be explained as the PSF, in the case of the fine sampling of projections, can be related to the beam profile using an Abel transform [46] which depicts the central region depression for a rectangular beam profile. For a 3 mm rectangular aperture of the detector, the beam profile as the convolution of Gaussian focal spot (rms width = 0.5 mm) and detector aperture has a wide flat region giving rise to central depression also known as the volcano effect [42].

The effect of varying the rms width of the focal spot on the PSF is shown in Figure 2.2.3. The PSF becomes narrower as the rms width of the focal spot is reduced from 2 mm to 0.5 mm. A further reduction from 0.5 mm to 0.125 mm does not produce significant change as the PSF is mainly determined by the detector aperture and sampling distance.

2.2.2 Reconstruction Algorithm

2.2.2.1 The Filter Function Design

The reconstruction process can produce different spatial resolution from the same projection data set depending upon the design of the filter impulse response. The difference between two commonly used filter functions is shown in Figure 2.2.4. The filter designed by Ramachandran and Lakshminarayana [23] is the same as filter function h_s in equation (2.1.15). Shepp and Logan [12] modified this filter function, which varies as sine instead of linear function of spatial frequency, to remove high frequency noise from the projections. The PSF, however, becomes wider for this modified filter since the high frequency part of the filter is effectively apodized.

2.2.2.2 Interpolation

The difference between the linear and nearest neighbor interpolation methods is shown in Figure 2.2.5. Ideally, the purpose of interpolation is to remove the frequencies above half the sampling rate from the filtered projection data, and to faithfully reproduce the spatial frequencies below half the sampling rate. Both the nearest neighbor and linear interpolation fail to accomplish this, however, a nearest neighbor interpolation function retains larger contributions from frequencies higher than half the sampling rate, and thus provides a more distorted PSF. Therefore, linear interpolation is more commonly used since it is computationally less expensive than higher order interpolations.

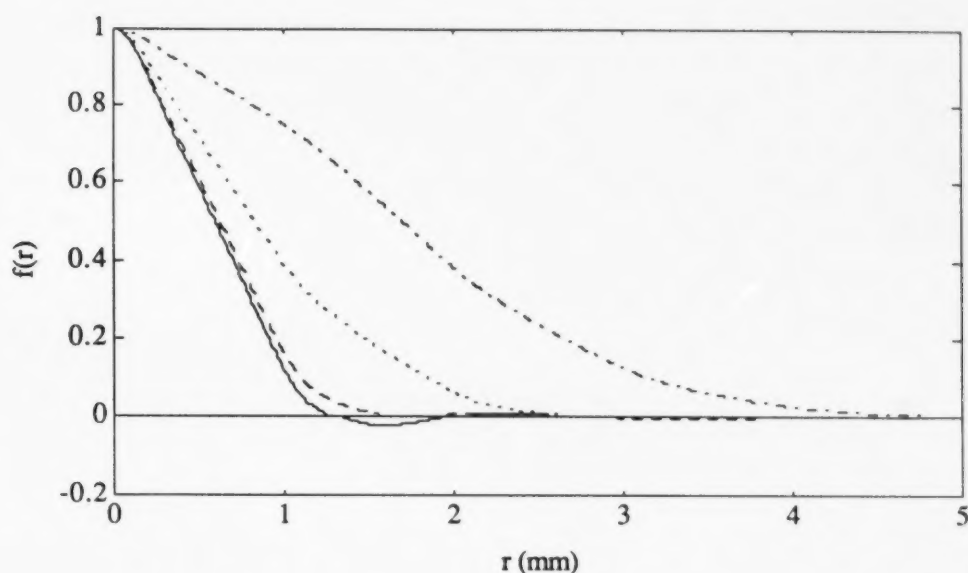


Figure 2.2.3: The PSFs for $r_0=1$ mm, $a=1$ mm, and $\Delta=0.125$ mm (solid), 0.5 mm (dashed), 1 mm (dotted) and 2 mm (dash-dotted). a = sampling distance, Δ = rms width of the focal spot intensity distribution, and r_0 = width of the detector sensitivity distribution.

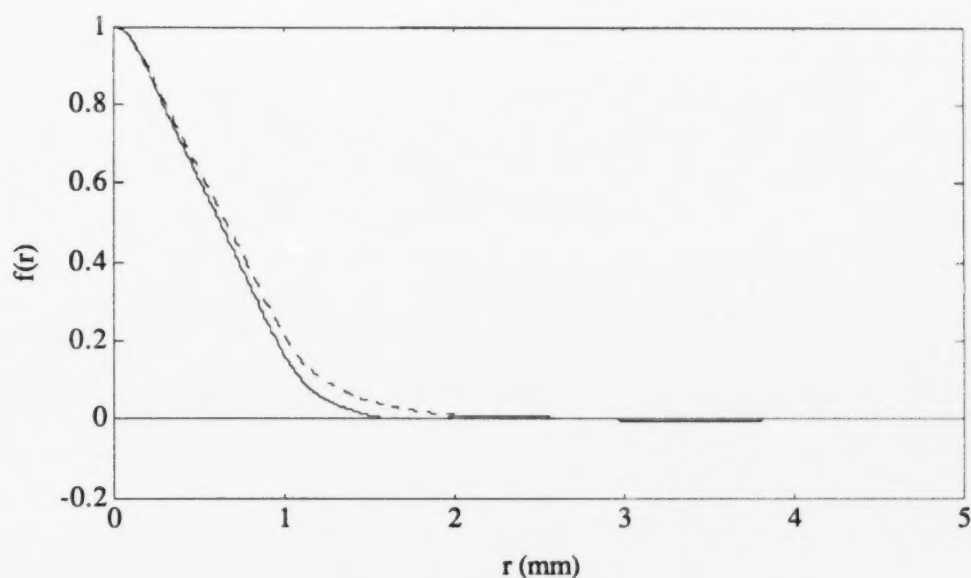


Figure 2.2.4: The PSFs for $r_0=1$ mm, $a=1$ mm, $\Delta=0.5$ mm, and Shepp and Logan (dashed) and Lakshminarayanan (solid) filter functions. a = sampling distance, Δ = rms width of the focal spot intensity distribution, and r_0 = width of the detector sensitivity distribution.

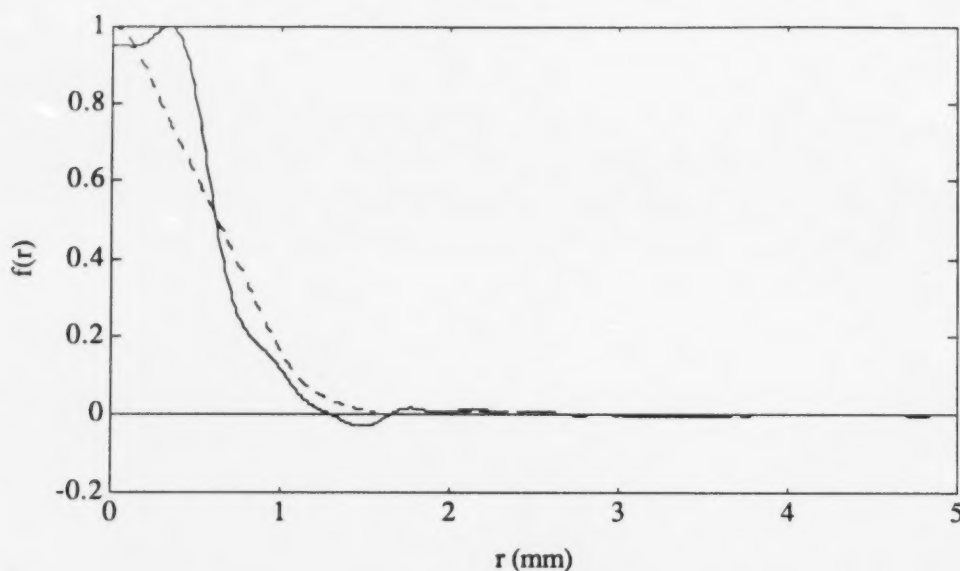


Figure 2.2.5: The PSFs for $r_0=1$ mm, $a=1$ mm, $\Delta=0.5$ mm, and nearest neighbor (solid) and linear (dashed) interpolation functions. a = sampling distance, Δ = rms width of the focal spot intensity distribution, and r_0 = width of the detector sensitivity distribution.

Theoretically, the PSF of a CT system with spatially invariant beam profile is spatially invariant. In practice, both the nearest neighbor and linear interpolations are not bandlimited making the PSF position dependent. Consider a point-object located at position (x_0, y_0) in Cartesian coordinates. If the measured projection of this point-object at an angle θ is sampled with a sampling distance a , the Fourier transform of the sampled projection is given by

$$P_{\text{samp}}(\omega, \theta) = \sum_{n=-\infty}^{\infty} \frac{\sin(\omega' r_0/2)}{\omega' r_0/2} \exp\left(-\frac{\omega'^2 \Delta^2}{2}\right) \exp[-i(x_0 \cos\theta + y_0 \sin\theta)\omega'] \quad (2.2.8a)$$

where

$$\omega' = \omega - \frac{2\pi n}{a} \quad (2.2.8b)$$

The PSF corresponding to this point-object can be reconstructed using the filtered backprojection method in Cartesian coordinates as follows

$$f(x, y) = \frac{1}{4\pi^2} \int_0^\pi d\theta \int_{-\infty}^{\infty} \exp[i\omega(x \cos\theta + y \sin\theta)] H_S(\omega) G(\omega) P_{\text{samp}}(\omega, \theta) d\omega \quad (2.2.9)$$

If the PSF was spatially invariant, then $f(x,y)$ would be a function of $(x-x_0)$ and $(y-y_0)$ which is true only for $n=0$ term in equation (2.2.8). Assuming that the projection data is sampled finely enough such that the replications of P_{samp} are not aliased, a bandlimited interpolation function $G(\omega)$ multiplying $P_{\text{samp}}(\omega,\theta)$ will retain only $n=0$ term, and remove all other terms for $n \neq 0$, resulting into a spatially invariant PSF. The nearest neighbor and linear interpolations are non-bandlimited and both retain some signal corresponding to the $n \neq 0$ terms, and the resulting PSF then becomes spatially variant [42]. In addition, circularly symmetric objects are distorted due to the 2-D aliasing resulting from backprojection of these non-bandlimited projections.

2.2.2.3 Pixel Size

In the previous section, the PSF was calculated under the assumption that the images are reconstructed on a continuous polar space. However, the reconstruction is generally carried out at a finite number of points in a 2-D Cartesian space on a square raster. Therefore, the spatial resolution is also affected by the reconstruction pixel size. The reconstruction pixel size is an independent parameter since the interpolation used before the backprojection makes the projections continuous, and a continuous reconstruction is theoretically possible. The spatial frequency information in these continuous filtered projections, however, is limited to half the sampling frequency of projection data. The signal allowed by using a non-ideal interpolation at higher than half the sampling frequency is not required. Therefore, it is common practice to make the reconstruction pixel size the same as the sampling distance of the projections such that the Nyquist rates of the reconstructed image and the measured projections are the same. This is useful since the Fourier transform of images reconstructed using backprojection of filtered projections can be computed by superposition of the 1-D Fourier transform slices of the filtered projections [45, 47]. A further reduction in the pixel size only increases the computational requirement of backprojection, and the storage requirement for the reconstructed image. In addition, the image contains useless and rather distorted information beyond half the sampling frequency of the projection data. The reconstruction pixel size must not be larger than the sampling distance of the projections to retain all the information in the projection data for the reconstructed image.

2.2.3 Number of Projections

The number of projections determine the area of the reconstructed image over which a given spatial resolution can uniformly be achieved [44, 48]. If v_0 is the maximum

allowable spatial frequency over the diameter D in the reconstructed image, then the required number of projections over the 180° range is given by πDv_0 . In case of 360° , the required number of projections can be intuitively determined as $2\pi Dv_0(1-\sin \psi/2)$ where ψ is the fan angle in a fan-beam geometry [43]. When calculating the required number of projections, care should be taken in evaluating the maximum allowable spatial frequency. Although, maximum allowable spatial frequency is given by $(1/2a)$ cycle/mm if both the sampling distance of the projection and reconstruction pixel size are equal to a , some high frequency components are completely attenuated due to finite size of beam profiles. A consequence of this is that an attempt to improve the spatial resolution beyond the reconstruction pixel size limit should be carefully analyzed since it not only requires the finer sampling of projection data but the number of projections should proportionately be increased. If the number of projections are not increased to specifications, then the improvement in the spatial resolution should only be carried out for the object lying in a circle of smaller radius than that of the reconstruction circle.

2.2.4 Noise

The discussion so far has neglected the effect of noise in projections resulting from various sources. The largest contribution to noise comes from the statistical nature of the collection of the finite number of photons. As a result of this, the spatially correlated noise is always present in the experimental CT images which affects the visual separation of small objects by decreasing, or distorting, the contrast resolution. The statistical noise in the projection data depends directly on the number of photons detected by a detector [11, 49], which in turn depends upon the number of photons emitted by the X-ray source in the direction of the detector, the attenuation properties of the underlying object, and the size of the detector collimator. If the spatial resolution is improved by making the sizes of the focal spot and detector aperture smaller, the noise in the projection data is increased since a fewer number of photons are emitted from the focal spot and a still fewer are detected due to smaller aperture at detector collimator. Similarly, decreasing the sampling distance for a fixed scan time provides fewer photons due to smaller accumulation (counting type detector) or integration (current integration mode) time in each sampling interval. Increasing the scan time results in an increased radiation dose delivered to the patient. Therefore, optimizing the CT system performance on the basis of dose, scan time, spatial resolution, noise and contrast resolution is a complex issue involving many inter-dependent parameters. In order to obtain a large number of finely sampled projections in a short scan time, the spatial resolution is compromised by using the larger sizes of the

detector aperture and focal spot.

In summary, the spatial resolution in a CT system is determined by many inter-related factors such as geometric parameters, reconstruction algorithm, scan time, contrast resolution and radiation dose. The theoretical spatial resolution is mainly determined by the size of the focal spot and detector aperture, and the sampling distance of the projections. For fixed sizes of the detector aperture and focal spot, the spatial resolution is improved by decreasing the sampling distance of the projections only up to a certain point (Figure 2.2.2). Beyond this point, decreasing the sampling distance does not provide any improvement in the spatial resolution. The theoretical limit to the spatial resolution is therefore provided by the finite size of the beam profile resulting from the finite sizes of the focal spot and detector aperture since these dimensions cannot be reduced without limit in order to retain meaningful statistics in the measured projections.

Restoration of CT images can be attempted to improve the spatial resolution beyond the limit imposed by the X-ray beam profiles. Some degradation due to increased noise in restored images is expected, however, radiation dose and the scan time are not affected since image restoration is carried out on the already collected data.

2.3 IMAGE RESTORATION

Image restoration is concerned with improving the quality of images which are degraded due to the instrumental imperfections in the imaging process. The most common form of the degradation is the linear spatial averaging produced by the imaging instrument. This may result from the relative motion between the object and the imaging instrument [53], astigmatism and curvature of the field [65], and finite size of the X-ray focal spot and detectors as, for example, in radiological imaging. The other form of degradation results from noise in the imaging process as, for example, in SPECT and PET [14, 66]. The spatial blur in some situations can also be non-linear as in photography, defined by the density vs. log exposure characteristic [53], and in CT resulting from the exponential law of X-ray attenuation [13]. In this section, we assume that the non-linear part of the spatial blur has been reduced to a linear blur either by an approximation or an appropriate transformation.

Let $f(x,y)$ and $g(x,y)$ be the original object and the degraded image, respectively, then the mathematical form of a model with linear spatial blur is given by

$$g(x,y) = \int \int f(x_1,y_1) h(x_1,y_1; x,y) dx_1 dy_1 + e(x,y) \quad (2.3.1)$$

where h is the PSF which gives the blurring function degrading $f(x,y)$, and $e(x,y)$ is the noise resulting from the imaging process. The PSF in equation (2.3.1) is, in general, a four-dimensional function for spatially variant blur. A spatially invariant PSF is independent of position (x,y) and depends only on the differences $(x-x_1,y-y_1)$. As a result, equation (2.3.1) can be written as a 2-D convolution

$$g(x,y) = \int \int f(x_1,y_1) h(x-x_1,y-y_1) dx_1 dy_1 + e(x,y) \quad (2.3.2)$$

If the PSF of the imaging system is such that the degraded image is a blurred version of the original object, then $h(x,y)$ represents a low pass spatial frequency filter. When considered in the 2-D spatial frequency domain, the magnitude of this filter decreases at high frequencies and may become zero at some frequencies.

Image processing using a digital computer can only be carried out when the recorded image is sampled on a finite size grid. Although the sampling can be carried out using any arbitrary geometry, the continuous function, $g(x,y)$, is commonly sampled regularly on a square grid. Let $g(j,k)$ describe the sampled form of the degraded image, on a N by N pixel grid, then equation (2.3.2) can be written as 2-D discrete convolution of the sequences $h(j,k)$ and $f(j,k)$, which describe the sampled PSF and the original image respectively.

$$g(j,k) = \sum_{m=1}^P \sum_{n=1}^P f(m,n) h(j-m,k-n) + e(j,k) \quad (2.3.3)$$

where it is assumed that P by P pixels of the original image are utilized to form the N by N degraded image, and if the PSF is M by M then $P=N+M-1$.

Let $g_e(j,k)$, $e_e(j,k)$, and $h_e(j,k)$ define the 2-D sequences of spatial extent P by P formed by padding zeroes around the sequences $g(j,k)$, $e(j,k)$, and $h(j,k)$, respectively. The system of equations (2.3.3) can then be written in a matrix form. Let P^2 -vectors F , G , and E define the original image $f(j,k)$, degraded image $g_e(j,k)$, and noise $e_e(j,k)$ by stacking these images row-wise (or column-wise). For example, let the first P elements of the vector G contain the pixels ($g_e(j,1)$, $j=1,P$) and second P elements contain the pixels

$(g_e(j,2), j=1, P)$ and so on. Then equation (2.3.3) can be written as

$$\mathbf{G} = \mathbf{H}\mathbf{F} + \mathbf{E} \quad (2.3.4)$$

where $\mathbf{H}$ is the blurring matrix of size P^2 by P^2 . It can be shown that $\mathbf{H}$ is a block-circulant matrix containing P^2 different partitions of size P by P [51, 52], so that $\mathbf{H}$ can be written as

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_P & \mathbf{H}_{P-1} & \dots & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_P & \dots & \mathbf{H}_3 \\ \mathbf{H}_3 & \mathbf{H}_2 & \mathbf{H}_1 & \dots & \mathbf{H}_4 \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{H}_P & \mathbf{H}_{P-1} & \mathbf{H}_{P-2} & \dots & \mathbf{H}_1 \end{bmatrix} \quad (2.3.5)$$

where each of the $\mathbf{H}_j$ is a circulant matrix having elements

$$\mathbf{H}_j = \begin{bmatrix} h_e(j,1) & h_e(j,P) & h_e(j,P-1) & \dots & h_e(j,2) \\ h_e(j,2) & h_e(j,1) & h_e(j,P) & \dots & h_e(j,3) \\ h_e(j,3) & h_e(j,2) & h_e(j,1) & \dots & h_e(j,4) \\ \dots & \dots & \dots & \dots & \dots \\ h_e(j,P) & h_e(j,P-1) & h_e(j,P-2) & \dots & h_e(j,1) \end{bmatrix} \quad (2.3.6)$$

The useful property of a block circulant matrix is that it can be diagonalized using the 2-D DFT [51, 52]. In other words, the matrix $\mathbf{H}$ can be written as

$$\mathbf{H} = \mathbf{W}\mathbf{D}\mathbf{W}^{-1} \quad (2.3.7)$$

where $\mathbf{D}$ is a diagonal matrix of eigenvalues with diagonal terms as the P^2 coefficients of 2-D DFT of the sequence $h_e(j,k)$. The matrix $\mathbf{W}$ is a P^2 by P^2 orthogonal matrix containing P^2 partitions of size P by P such that

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}(1,1) & \mathbf{W}(1,2) & \mathbf{W}(1,3) & \dots & \mathbf{W}(1,P) \\ \mathbf{W}(2,1) & \mathbf{W}(2,2) & \mathbf{W}(2,3) & \dots & \mathbf{W}(2,P) \\ \mathbf{W}(3,1) & \mathbf{W}(3,2) & \mathbf{W}(3,3) & \dots & \mathbf{W}(3,P) \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{W}(P,1) & \mathbf{W}(P,2) & \mathbf{W}(P,3) & \dots & \mathbf{W}(P,P) \end{bmatrix} \quad (2.3.8)$$

where the mn^{th} matrix $\mathbf{W}(m,n)$ is given as the following

$$W(m,n) = \exp(i2\pi mn/P) \begin{bmatrix} w(1,1) & w(1,2) & w(1,3) & \dots & w(1,P) \\ w(2,1) & w(2,2) & w(2,3) & \dots & w(2,P) \\ w(3,1) & w(3,2) & w(3,3) & \dots & w(3,P) \\ \dots & \dots & \dots & \dots & \dots \\ w(P,1) & w(P,2) & w(P,3) & \dots & w(P,P) \end{bmatrix} \quad (2.3.9)$$

where the scalar element $w(j,k)$ is given by

$$w(j,k) = \exp(i2\pi jk/P) \quad (2.3.10)$$

The need for extending the sequences can now be stated as follows. It is known that multiplying the 2-D DFTs of two sequences is the same as a 2-D circular convolution between them in the spatial domain. Therefore, in order to make the linear and circular convolutions identical within a P by P window, the length P should be chosen as $P \geq N+M-1$. This will result in H being block-circulant which can be diagonalized as discussed previously. Equation (2.3.4) can also be written for spatially variant PSF. However, the matrix H is no longer block-circulant. The diagonalization can be carried out either by direct inversion, or iteratively which would require storage of a four dimensional PSF, $h(j,k; m,n)$. In either the spatial variant or invariant cases, the restoration problem is to determine the vector F when the vector G , and the matrix H , are known. Since most of the classical restoration techniques are designed for a spatially invariant PSF, the discussion in the following section is restricted to such PSFs only.

The image restoration problem is usually either ill-posed or ill-conditioned since some of the eigenvalues of the matrix H are either zero or very small. Therefore, a direct solution of equation (2.3.4) is either not unique or undesirable due to the excessive amplification of noise, as in the case of the inverse filter. Regularization is a technique for obtaining a satisfactory solution for ill-posed and ill-conditioned problems by utilizing the other *a priori* constraints regarding the physical object in an image. There are two types of constraints, deterministic and statistical, which have been used in image restoration. The deterministic constraints such as non-negativity, finite spatial support, bandlimitedness, bounded pixel values etc., are more commonly utilized in an iterative procedure for image restoration using the theory of convex sets [33]. In the following sections, regularization of the restoration problem using probabilistic constraints is discussed in terms of Bayes theorem. Using this formulation, a solution of the restoration problem is determined by maximizing the *a posteriori* (MAP) probability of obtaining a restored distribution, calculated from an assumed *a priori* probability distribution and conditional probability of

obtaining the blurred image calculated from the noise properties. This formulation is then used to derive the classical restoration approaches such as the inverse filter, the Wiener filter and constrained least squares, and the more sophisticated maximum entropy (ME) approach. The approach chosen depends upon the specific assumption made to calculate the *a priori* probability of the original image within the ensemble of objects.

2.3.1 *A Posteriori* Probability and Bayes Theorem

A number of restoration techniques such as the inverse filter, the Wiener filter and the constrained least squares discussed in the literature [36, 52] seldom make a connection between the Fourier domain formulation and Bayes theorem. It will be shown in the forthcoming sections that these technique are all maximum *a posteriori* (MAP) solutions [36, 51] of equation (2.3.4), and its solution in the 2-D Fourier domain results from the specific assumptions made about the *a priori* probability distribution of the original image and the conditional probability distribution of the blurred image on the original image. Consider that both **F** and **E** are single realizations out of their respective randomly occurring ensembles. The image noise, in general, is a random process whose properties can be derived from the physical characteristics of the imaging process, in terms of the probability distribution function (PDF), **P(E)**. The original image, on the other hand, is deterministic in nature. By considering that a particular experiment is used to image only a single class of objects, then the vector **F** can be considered as one realization from this class. The statistical properties of the class of objects in terms of the PDF, **P(F)**, may or may not be known *a priori*. The conditional probability of observing **F** given **G**, denoted by **P(F/G)**, can be calculated using the Bayes theorem

$$P(F/G) = \frac{P(G/F)P(F)}{P(G)} \quad (2.3.11)$$

where **P(G/F)** is the conditional probability of observing **G** given one realization **F** of the original image from the ensemble, and determined from the PDF of the noise since **H** is assumed to be known as a deterministic mapping.

In the classical techniques, it is assumed that both **P(E)** and **P(F)** are Gaussian distributions.

$$P(F) = k_1 \exp [-(F-F_m)^T R_{ff}^{-1} (F-F_m)] \quad (2.3.12)$$

$$P(E) = k_2 \exp [-E^T R_{ee}^{-1} E] \quad (2.3.13)$$

where k_1 and k_2 are normalization constants, and matrices R_{ff} and R_{ee} are the covariance matrices of the original image and noise processes respectively. The vector F_m defines the ensemble mean of F over the class of the original object, and the noise is assumed to be of zero-mean. In equations (2.3.12) and (2.3.13), $()^T$ denotes the transpose of the matrix. Assuming $K=k_1k_2/P(G)$ is another constant since $P(G)$ is independent of F , and using equations (2.3.12) and (2.3.13), and since $P(G/F)=P(E)$, equation (2.3.11) can be modified to the following.

$$P(F/G) = K \exp [-(F-F_m)^T R_{ff}^{-1} (F-F_m) - E^T R_{ee}^{-1} E] \quad (2.3.14)$$

Now taking the logarithm on both sides of equation (2.3.14) one gets

$$\ln P(F/G) = \ln K - (F-F_m)^T R_{ff}^{-1} (F-F_m) - E^T R_{ee}^{-1} E \quad (2.3.15)$$

A MAP solution can be determined by maximizing $\log P(F/G)$ since the logarithm is a monotonically increasing function of its argument. The term $\ln K$ can be ignored because it is a constant independent of F , and the maximization problem can be changed to a minimization problem by removing the negative sign on the second term of equation (2.3.15). In addition, the vector E can be replaced using equation (2.3.4). Therefore, the MAP restoration is identical to that obtained by minimizing the function

$$J(F) = (F-F_m)^T R_{ff}^{-1} (F-F_m) + (G-HF)^T R_{ee}^{-1} (G-HF) \quad (2.3.16)$$

2.3.1.1 Inverse Filter

In case of an inverse filter, the *a priori* probability, $P(F)$, is assumed to be constant irrespective of the original image, as a result, the first term on the right hand side of equation (2.3.16) becomes a constant independent of F . In addition, the noise in the restoration model of equation (2.3.3) is assumed to be uncorrelated such that R_{ee} is a unit diagonal matrix. Modifying equation (2.3.16) accordingly, and then taking the derivatives with respect to the elements of vector F results in the following system of equations.

$$H^T H F' = H^T G \quad (2.3.17)$$

where F' denotes the vector corresponding to the restored image.

Substituting the diagonalized form of the matrix H , and noting that $W^T W = I$; equation (2.3.17) is modified to the following

$$\mathbf{W}\mathbf{D}^*\mathbf{D}\mathbf{W}^{-1}\mathbf{F}' = \mathbf{W}\mathbf{D}^*\mathbf{W}^{-1}\mathbf{G} \quad (2.3.18)$$

It should be noted that multiplying $\mathbf{W}^{-1}$ with the vectors $\mathbf{G}$ and $\mathbf{F}'$ is the same as computing the 2-D DFTs, $G(k_1, k_2)$ and $F'(k_1, k_2)$, respectively, for the blurred image $g_e(j, k)$ and the resulting restoration $f'(j, k)$. Also the diagonal matrix $\mathbf{D}$ has elements corresponding to the 2-D DFT of the PSF, $h_e(j, k)$, and $\mathbf{D}^*$ is the complex conjugate of $\mathbf{D}$. Denoting the 2-D DFT of $h_e(j, k)$ as $H(k_1, k_2)$, and pre-multiplying both sides of equation (2.3.18) by $\mathbf{W}^{-1}$, the Fourier domain form of this equation can be written as

$$F'(k_1, k_2) = \frac{H^*(k_1, k_2)G(k_1, k_2)}{H^*(k_1, k_2)H(k_1, k_2)} \quad (2.3.19)$$

The factor $H^*(k_1, k_2)$ can be cancelled from the numerator and the denominator, which results in the classical inverse filter restoration, given by

$$F'(k_1, k_2) = \frac{G(k_1, k_2)}{H(k_1, k_2)} \quad (2.3.20)$$

As seen in equation (2.3.20), the inverse filter is intuitively justified; the frequency components, which are de-emphasized by $H(k_1, k_2)$, are re-emphasized by $1/H(k_1, k_2)$. However, for the zeroes of $H(k_1, k_2)$, there is no solution since those components in $G(k_1, k_2)$ have been removed altogether. The presence of noise further compounds the problem near zeroes of $H(k_1, k_2)$ since small values in the denominator magnify the noise. Even if there are no zeroes in $H(k_1, k_2)$, it will always have small values at high frequencies, and these values will magnify the noise in the restored image. For these reasons, the inverse filter is never used in practice except when realized iteratively along with some deterministic constraints [50].

2.3.1.2 Wiener Filter

This procedure assumes that the random fields corresponding to the noise and the original object are stationary and are samples from Gaussian PDFs. In addition, the ensemble mean of either the noise or the original image is assumed to be zero. The Wiener filter can then be derived from equation (2.3.16) by equating its derivative with respect to the elements of $\mathbf{F}$ to zeroes, which results in the following after re-grouping the terms corresponding to $\mathbf{G}$ on the right hand side

$$(\mathbf{R}_{ff}^{-1} + \mathbf{H}^T \mathbf{R}_{ee}^{-1} \mathbf{H}) \mathbf{F}' - \mathbf{R}_{ff}^{-1} \mathbf{F}_m = \mathbf{H}^T \mathbf{R}_{ee}^{-1} \mathbf{G} \quad (2.3.21)$$

where $\mathbf{F}'$ is the vector representing the solution of equation (2.3.21).

In case of the stationary random fields for the noise and the original object, the covariance matrices $\mathbf{R}_{ff}$ and $\mathbf{R}_{ee}$ describe the position invariant covariance functions. In a manner similar to the block-circulant representation of $\mathbf{H}$ corresponding to the extended sequence $h_e(j,k)$, the matrices $\mathbf{R}_{ff}$ and $\mathbf{R}_{ee}$ can be formed as block-circulant after suitably extending the corresponding covariance functions. Therefore, both $\mathbf{R}_{ff}$ and $\mathbf{R}_{ee}$ can be diagonalized such that

$$\mathbf{R}_{ff} = \mathbf{W}\mathbf{S}_f\mathbf{W}^{-1} \quad (2.3.22)$$

and

$$\mathbf{R}_{ee} = \mathbf{W}\mathbf{S}_e\mathbf{W}^{-1} \quad (2.3.23)$$

where $\mathbf{S}_f$ and $\mathbf{S}_e$ are the diagonal matrices with elements equal to the P^2 -values of the 2-D DFTs of the covariance functions of the original image and noise, respectively. The orthogonal matrix $\mathbf{W}$ is the same as that defined previously containing the complex exponential coefficients in the DFT expansion. Equation (2.3.21) can now be modified to

$$\mathbf{W}(\mathbf{S}_f^{-1} + \mathbf{D}^* \mathbf{S}_e^{-1} \mathbf{D}) \mathbf{W}^{-1} \mathbf{F}' - \mathbf{W} \mathbf{S}_f^{-1} \mathbf{W}^{-1} \mathbf{F}_m = \mathbf{W} \mathbf{D}^* \mathbf{S}_e^{-1} \mathbf{W}^{-1} \mathbf{G} \quad (2.3.24)$$

After pre-multiplying each term on both sides of equation (2.3.24) by $\mathbf{W}^{-1}$, and realizing that operations $\mathbf{W}^{-1} \mathbf{F}'$, $\mathbf{W}^{-1} \mathbf{G}$, and $\mathbf{W}^{-1} \mathbf{F}_m$ are the same as taking 2-D DFTs of the restored image $f'(j,k)$, the blurred image $g_e(j,k)$ and the *a priori* mean image, respectively, equation (2.3.24) can be written in terms of DFT variables k_1 and k_2 .

$$\left[\frac{S_e(k_1, k_2)}{S_f(k_1, k_2)} + H^*(k_1, k_2) H(k_1, k_2) \right] F'(k_1, k_2) - \frac{S_e(k_1, k_2)}{S_f(k_1, k_2)} F_m(k_1, k_2) = H^*(k_1, k_2) G(k_1, k_2) \quad (2.3.25)$$

where S_e , S_f , and F_m are the NPS, the 2-D DFT of the covariance function of the original image, and the 2-D DFT of the mean image, respectively.

If the random field corresponding to the original object is stationary, then the second term on the left-hand-side of equation (2.3.25) can be ignored since it only affects the mean intensity in the restored image. Therefore, the Wiener filter restoration in the 2-D DFT domain can be written as

$$F'(k_1, k_2) = \frac{H^*(k_1, k_2) G(k_1, k_2)}{H^*(k_1, k_2) H(k_1, k_2) + \frac{S_e(k_1, k_2)}{S_f(k_1, k_2)}} \quad (2.3.26)$$

It can be seen that the Wiener filter reduces to an inverse filter, in the case when no noise is present in the blurred image. The second term in the denominator results in the Wiener filter being different from the inverse filter in that denominator does not reduce to zero when $H(k_1, k_2)$ becomes zero. The fraction S_e/S_f guides the filter such that more weight is given to this term in the case of noisy images which results into a smoother restoration and vice versa. The *a priori* knowledge of power spectrum S_f of the original image also guides the filter, for example, if it is known that the original image contains zero signal beyond a specified spatial frequency, the resulting Wiener restoration would also be zero.

The derivation for the Wiener filter can also be carried out by minimizing the mean-squared-error, $E\{(F-LG)^T(F-LG)\}$, where E is the expectation, with respect to the restoration filter, L , [51]. The major problem with the Wiener filter is that the power spectra of the original image and the noise must be known *a priori*. Such information, in particular S_f , is rarely available and any other assumptions cause the results to deviate from the optimal.

2.3.1.3 Constrained Least-squares (CLS) Restoration

Although the Wiener filter is an optimal linear estimate [36], in the mean-squared-error sense, it requires the knowledge of the power spectra of the noise and original object. When this information is not available, image restoration can be carried out using the CLS technique. The CLS technique can be formulated as the following constrained minimization problem [36].

minimize

$$(QF)^T(QF)$$

subject to

(2.3.27)

$$(G-HF)^T(G-HF) \leq e^2$$

where matrix Q describes an operator which measures the roughness of the image, and e^2 depends upon the norm of noise in the blurred image. The most common form of operator described by Q is a digital approximation to the second derivative (Laplacian) of 2-D functions. The second derivative for an image can be calculated by convolving the image with a high-pass filter sequence whose coefficients are provided by this digital approximation. Therefore, by appropriately extending this sequence with zeroes, the matrix

Q can also be made block-circulant.

The derivation of a CLS filter can be carried out by using the Lagrange multiplier. Using the Lagrange multiplier γ , the solution can be obtained by minimizing $J(F)$ defined as the following.

$$J(F) = \gamma(QF)^T(QF) + (G-HF)^T(G-HF) \quad (2.3.28)$$

After equating the derivative of J with respect to F to zero and rearranging, the restored image F' can be obtained by solving the following equations.

$$(\gamma Q^T Q + H^T H)F' = H^T G \quad (2.2.29)$$

A value of the parameter γ is determined which makes the solution vector F' satisfy the constraint in equation (2.3.27). It should be remembered that the operator Q is completely arbitrary and its optimal characteristics can only be determined experimentally [66].

In this section, a derivation using equation (2.3.21) will be also given. In equation (2.3.21), since each of the matrices can be diagonalized using the 2-D DFT, the matrix multiplications commute. Therefore, by multiplying both sides of equation (2.3.21) by R_{ee} and dropping F_m term since it only affects the mean intensity, the resulting equation can be written as

$$(R_{ee}R_{ff}^{-1} + H^T H)F' = H^T G \quad (2.3.30)$$

It is obvious that equations (2.2.29) and (2.3.30) are identical if $R_{ee}R_{ff}^{-1} = \gamma Q^T Q$. The CLS filter is thus related to the Wiener filter in that the roughness operator matrix Q is related to the power spectra of the noise and the original image. Therefore, the CLS filter also has the specific formulation of restoration using the Bayes theorem.

Using the diagonalization of the matrices in equation (2.3.29), the solution $F'(k_1, k_2)$ can be written in terms of the 2-D DFT $Q(k_1, k_2)$ of the roughness operator such that

$$F'(k_1, k_2) = \frac{H^*(k_1, k_2)G(k_1, k_2)}{\gamma Q^*(k_1, k_2)Q(k_1, k_2) + H^*(k_1, k_2)H(k_1, k_2)} \quad (2.3.31)$$

The Bayesian formulation of the CLS method is applicable for non-stationary random fields describing the original object and noise. In addition, spatially variant and

nonlinear restoration can also be carried out. In these cases, the resulting matrices are no longer block-circulant and the diagonalization is carried out iteratively. Nonlinear restoration is carried out using the Newton-Raphson technique for solving the nonlinear system of simultaneous equations. Nonstationary restoration based upon the nonstationary variance calculated from the blurred image has been reported in the case of motion blur and significantly reduces the ringing artifacts in images [57].

2.3.1.4 Maximum Entropy Restoration

In the CLS and Wiener filter methods, the regularization of the restoration problem is performed by assuming *a priori* Gaussian distributions without providing any justification. Theoretically, this choice of distribution is completely arbitrary as is the fact that the roughness operator Q correctly describes the underlying object's covariance function. In the ME approach [58] these assumptions regarding the original image, and the statistics of the measurement noise [59], are avoided. The *a priori* distribution $P(F)$, in the case of ME method, is derived as follows. Let $f_1, f_2, \dots, f_P$ be the elements of the vector F for the original image. Let O be the total intensity in the original image such that

$$O = \sum_{k=1}^P f_k \quad (2.3.32)$$

Let us consider an experiment in which a unit intensity is assigned to a pixel location in random fashion until the total intensity is distributed, and assume that this assignment is independent for all the pixels. Using this process, the number of different ways, W , the total intensity O can be distributed to form the intensity pattern $\{f_1, f_2, \dots, f_P\}$ is given by the multinomial coefficient

$$W = \frac{O!}{f_1! f_2! \dots f_P!} \quad (2.3.33)$$

For a given O and P , an image vector most likely to occur is the one which maximizes W subject to the constraint that equation (2.3.4) is satisfied in some sense. Instead of maximizing W , it is easier to maximize $O^{-1} \log W$ since then, under Stirling approximation and as P approaches ∞ ,

$$O^{-1} \log W \approx - \sum_{k=1}^P p_k \log p_k \quad (2.3.34)$$

where $p_k = f_k/O$ [58].

Equation (2.3.33) gives $P(F)$ and equation (2.3.34) gives logarithm of the *a priori* probability [58], therefore

$$P(F) \propto \exp \left(-O \sum_{k=1}^P p_k \log p_k \right) \quad (2.3.35)$$

Let us assume that the noise is known to be a zero mean, uncorrelated Gaussian process of variance σ^2 , although a similar treatment of the noise can be carried out [59] if the noise properties are not unknown. The log *a posteriori* probability of equation (2.3.15) can be written as

$$\log P(F/G) = \log K - O \sum_{k=1}^P f_k/O \log f_k/O - E^T R_{ee}^{-1} E \quad (2.3.36)$$

Since R_{ee} is a diagonal matrix with all the elements equal to σ^2 , the MAP solution is obtained by maximizing the following function.

$$J(F) = - \sum_{k=1}^P f_k/O \log f_k/O - \gamma (HF-G)^T (HF-G) \quad (2.3.37)$$

where $\gamma = 1/O\sigma^2$, and the constant term $\log K$ is ignored.

The first term in equation (2.3.37) is the entropy of the original image. The MAP formulation is thus the same as the ME approach which states that out of all the solutions which satisfy equation (2.3.4) in a least-squares sense, a solution which has the largest entropy will maximize the number of ways the intensity O could be distributed in the image. It should be remembered that this does not make any assumption regarding the *a priori* distribution other than the fact that the intensity distribution in the pixel locations is statistically independent.

The ME approach can also be understood in terms of the smoothness and spurious structures in the image. The ME principle, in general, states that the image generated by this method will be the smoothest image, or the image with no structures for which there is no account provided in the equation (2.3.4) and the other knowledge regarding the object.

Equation (2.3.37) is a unconstrained optimization problem with a nonlinear cost function. The solution of this problem is extremely sensitive to the type of algorithm used. As a result, a number of different methods have been used [60, 61]. The method based upon the local quadratic approximation and distance constraint [61], and the technique

based upon the solution of differential equation [60] are most well known.

Despite its popularity in image restoration and reconstruction, the ME method has not found applications in real world image restoration problems. The practical reason for this is the computational requirement for this method; the best optimized ME algorithms are 100 or more times slower than the classical approaches. The other criticism of the ME method is that it is derived using the statistical independence of pixel locations. In many practical applications, the PSF is unknown and classical methods can be modified to deal with simultaneously identifying the PSF and performing restoration [64] whereas no such formulation of ME has been reported.

2.3.2 Other Methods

The methods based upon MAP are only a few approaches which have been reported in the literature. The regularized imaged restoration is a general approach, which does not require references to Bayes theorem [54, 55, 56]. There are a number of other techniques for image restoration such as Kalman filtering [36], projection onto the convex sets (POCS) [62], and neural networks. The Kalman filter is a linear least-mean-squared error method, similar to the Wiener filter, but which is based upon the Markov field representation of the original image. It is applied recursively in the spatial domain. The POCS technique is based upon determining the intersection of the convex sets, which describe each kind of constraint available regarding the original image, by recursively projecting the image vector on each of the convex sets. In the neural network, the neural representation of the image is sought which provides least energy of the network [63]. This approach may be implementable using a special purpose hardware design. There are techniques which represent the image as an autoregressive process and the known or unknown PSF as a moving average process, and are capable of identifying the PSF for an arbitrary image when a model for the physical experiment is unknown, and the parametric approaches to identify the PSF remain unsuccessful [64]. Although the image restoration problem is always regularized using either the Bayes theorem or the other techniques, an inverse filter, if it exists, is the only exact solution for this problem. All other approaches find a suitable restored image when the inverse filter fails to provide a satisfactory restoration. Therefore, the selection of a restoration method should be based upon the properties of the PSF. If some of the eigenvalues of PSF matrix approach zero, or are very small, regularization should be considered. For example, it is fruitless to restore the image blurred by uniform motion using the inverse filter.

In order to avoid the undue amplification of noise in the restored image, the regularization using the NPS should be considered, as in the Wiener filter, since the eigen-spectrum of the noise covariance matrix falls very slowly to zero compared to those of the practical PSF matrices. In addition, one should seek to utilize as much information about the physical experiment and process under study as possible since the iterative procedures can always be mixed with POCS to reduce artifacts in the image.

Many imaging experiments result in nonlinear spatial blurring and nonstationary PSF and noise characteristics. Although a MAP formulation can always be made, this approach would often require an iterative solution of the nonlinear system of equations with dense and large coefficient matrix which is not block circulant. In these situations, simplifications regarding linearization, position invariant approximations to the PSF etc. should be explored before considering a general purpose MAP solution.

3. FORMULATION OF IMAGE RESTORATION IN COMPUTED TOMOGRAPHY

In the previous chapter, a critical analysis of the spatial resolution in CT indicated that the finite sizes of the focal spot and detector aperture provide an inherent limit to the achievable spatial resolution in reconstructed images. The finite sizes of the focal spot and detector aperture result in the finite size of the X-ray beams. The image reconstruction methods described in section 2.1 do not incorporate the finite size of the X-ray beams in the reconstruction procedure. The derivation of the filtered backprojection method is based upon the assumption that a CT system collects projections using the infinitesimally narrow X-ray beams. Image restoration methods incorporating the finite size of the X-ray beams can be used to improve the spatial resolution. In this chapter, a formulation of the image restoration method in CT is discussed in two ways, from the projection data and by using the reconstructed CT images. Simple restoration techniques applied to both kinds of formulations are discussed for the case of an idealized CT system with a spatially invariant PSF. Image restoration is applied to the mathematically generated objects for which the projection data at each angle, using a finite size Gaussian beam profile, is computed by assuming that the CT system contains a single detector. The performance of the restored images is evaluated by comparison with those obtained by a filtered backprojection method.

3.1 FORMULATION OF IMAGE RESTORATION

The image restoration problem in CT can be formulated in two ways. In the first approach, the solution is obtained directly from the projection data. Since the geometries of the X-ray beam and the scanner define the transformation of the object distribution from Cartesian coordinates into the projection space, this method requires that the X-ray beam profiles corresponding to each detector in the system be known. In the second approach, images reconstructed by using the filtered backprojection, which are blurred due to the finite size of the X-ray beam, are restored using the PSF and the NPS.

3.1.1 Restoration from Projection Data

In this section, a linear relationship between the projection data and the attenuation coefficient distribution of an object is derived with certain approximations, and a solution of this linear system is then considered.

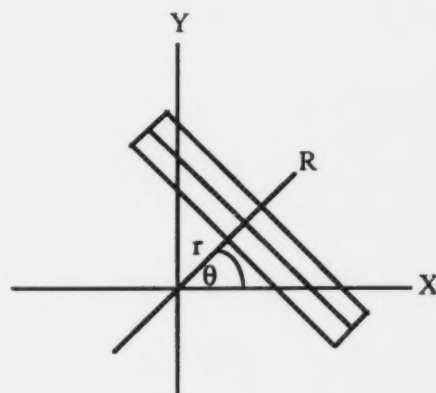


Figure 3.1.1: Data collection geometry of an idealized CT system, the narrow strip describes the X-ray beam profile.

The data collection geometry of a simple translate-rotate CT scanner is shown in Figure 3.1.1. Let a 2-D real valued function $f(x,y)$ describe the distribution of linear attenuation coefficient of an object in the (x,y) plane. Let $K_0(r)$ describe the distribution of photon intensity in the X-ray beam profile (Figure 3.1.1) and assume that

$$K_0(r) = \frac{N_0}{\sqrt{2\pi}\Delta} \exp\left(-\frac{r^2}{2\Delta^2}\right) \quad (3.1.1)$$

where N_0 is the total number of photons in the beam and Δ is the root-mean-squared (rms) width of the Gaussian beam profile. As indicated in section (2.2), the X-beam profile is determined from the focal spot intensity and detector sensitivity distributions. Since detectors in a CT system are located farther away from the center of the scan field, the X-beam profile is mainly determined by the focal spot intensity distribution. The focal spot intensity distribution is, generally, a Gaussian function since photon emission from the focal spot is a statistical process.

The total number of photons $I(r, \theta)$ in the beam after passing through the object is given by

$$I(r, \theta) = \int_{-\infty}^{\infty} K_0(r') \exp[-p_1(r-r', \theta)] dr' \quad (3.1.2)$$

where $p_1(r, \theta)$ is the line integral of the function $f(x,y)$ as given in equations (2.1.1 and 2.1.2). The line integral of this function is the same as the total attenuation provided by the object along that line.

Equation (3.1.2) results from the assumption that the X-ray beam is made up of parallel rays. The number of photons in each of these rays is attenuated according to the exponent of the line integral. After the beam has passed through the object, an ideal photon detector measures the total number of photons $I(r,\theta)$ remaining in the beam. The size of the X-ray beam is very small compared to the scan field as indicated by the narrow strip in Figure 3.1.1. Therefore, the limits of integration provided in equation (3.1.2) are only theoretical.

Since the total number of photons in the beam without attenuation is N_0 , the projections $p_s(r,\theta)$ of the object measured by using the finite size beam is given as the log ratio of N_0 and $I(r,\theta)$; the relationship between $p_s(r,\theta)$ and $p_l(r,\theta)$ is thus strictly non-linear. To implement a linear restoration, the relationship between $p_s(r,\theta)$ and $p_l(r,\theta)$ must be linear. By assuming that a projection $p_l(r,\theta)$ does not change significantly within the beam width, the exponent of the variations $p_l'(r,\theta)$ around the mean $\mu(r,\theta)$ can be expanded as the first two terms of a Taylor series such that

$$I(r,\theta) = \int_{-\infty}^{\infty} K_0(r') \exp[-\mu(r,\theta)][1-p_l'(r-r',\theta)]dr' \quad (3.1.3)$$

which can be modified to the following

$$I(r,\theta) = N_0 \exp[-\mu(r,\theta)] \left[1 - \frac{1}{N_0} \int K_0(r') p_l'(r-r',\theta) dr'\right] \quad (3.1.4)$$

and then

$$p_s(r,\theta) = \text{Ln}[N_0] - \text{Ln}[I(r,\theta)] = \mu(r,\theta) + \text{Ln} \left[1 - \frac{1}{N_0} \int K_0(r') p_l'(r-r',\theta) dr'\right]^{-1} \quad (3.1.5)$$

where Ln denotes the natural logarithm.

Although the original object may contain fine details with high contrast, in a projection of line integrals $p_l(r,\theta)$ these details are seen as small variations around the mean value $\mu(r,\theta)$ within the beam width. Therefore, by noting that for a small value of α , $\text{Log}(1/1-\alpha)$ is approximately equal to α , equation (3.1.5) can be modified to the following

$$p_s(r, \theta) = \frac{1}{N_0} \int_{-\infty}^{\infty} K_0(r') p_l(r-r', \theta) dr' \quad (3.1.6)$$

Since the relationship between $f(x, y)$ and $p_l(r, \theta)$ is linear, the relationship between $p_s(r, \theta)$ and $f(x, y)$ is also linear under the assumptions made in equation (3.1.6). The parallel beam profile $K_0(r)$ is independent of the perpendicular direction to r . Therefore, using equation (2.1.2), equation (3.1.6) can be written in terms of the function $f(x, y)$ as

$$p_s(r, \theta) = \frac{1}{N_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) K_0(r-x \cos \theta - y \sin \theta) dx dy \quad (3.1.7)$$

Equation (3.1.7) is derived by substituting $p_l(r-r', \theta)$, which is given by equation (2.1.2), into equation (3.1.6). The integration over the dummy variable r' vanishes due to $\delta(x \cos \theta + y \sin \theta - r + r')$.

Let the N^2 -vector $\mathbf{F}$ contain the column-wise (or row-wise) arranged values of the sampled $f(x, y)$ on an N -by- N pixel grid, and a MN -vector $\mathbf{P}$ contain M sampled projections each having N elements. The algebraic relationship between $\mathbf{F}$ and $\mathbf{P}$ in the matrix form is then given by

$$\mathbf{P} = \mathbf{K} \mathbf{F} \quad (3.1.8)$$

The matrix $\mathbf{K}$ transforms $\mathbf{F}$ to $\mathbf{P}$ according to equation (3.1.7), and the number of columns in the matrix $\mathbf{K}$ is the same as the length of vector $\mathbf{F}$. Therefore, each row of $\mathbf{K}$ provides the factors which weight the elements of $\mathbf{F}$ in order to obtain one element of $\mathbf{P}$. It is evident from equation (3.1.7) that these factors are determined by the beam profile. Since the width of the beam profile is very small compared to the diameter of the scan field, most of these factors are zero making $\mathbf{K}$ very sparse.

3.1.1.1 Noise

A number of different sources introduce random uncertainty to the measured photon intensity, however, the major portion of this noise results from the statistical nature of photon counting. Since the photon detection phenomenon is discrete, the law of probability is governed by the Poisson PDF. Therefore, the transmitted photon intensity at each position (r, θ) is a random variable with both mean and variance equal to $I(r, \theta)$. However, due to the spatial averaging which occurs over the beam profile, and because $I(r, \theta)$ is usually a large number, the Poisson PDF can be approximated as a Gaussian PDF of mean

and variance both equal to $I(r,\theta)$.

As a result of the noise in the photon counting, the projection data computed from the transmitted photon intensity is also noisy. Let a MN-vector $\mathbf{E}$ contain the zero-mean, Gaussian noise $e(r,\theta)$ in the projection data vector $\mathbf{P}$. The variance of this noise process is computed from the transmitted photon intensity.

In theory, both N_0 and $I(r,\theta)$ are random variables, however, N_0 can be considered as a deterministic constant for the purpose of computing the noise variance in the projection data. Firstly, this is because N_0 is generally much larger than $I(r,\theta)$. Secondly, the value of N_0 is calculated by performing a CT scan with no object placed in the scan field, and adding the measured photon intensity from all angles. This significantly reduces the error due to statistical counting. The measured, noisy projections $p_{sn}(r,\theta)$ are calculated as the log ratio of N_0 and measured photon intensity $[I(r,\theta)+i(r,\theta)]$ as follows

$$p_{sn}(r,\theta) = \text{Ln } N_0 - \text{Ln } [I(r,\theta)+i(r,\theta)] \quad (3.1.9)$$

where $i(r,\theta)$ is a Gaussian random number with mean equal to zero and variance equal to $I(r,\theta)$, as a statistical part of the transmitted photon intensity measured by the detector.

In the case of high signal to noise ratios, the random variable $i(r,\theta)$ is always very small compared to $I(r,\theta)$, therefore

$$p_{sn}(r,\theta) = \text{Ln } N_0 - \text{Ln } [I(r,\theta)] + \frac{i(r,\theta)}{I(r,\theta)} \quad (3.1.10)$$

Using equation (3.1.5) this can be written as

$$p_{sn}(r,\theta) = p_s(r,\theta) + e(r,\theta) \quad (3.1.11)$$

where $e(r,\theta)$ is a random variable with mean equal to zero and variance equal to $1/I(r,\theta)$.

Therefore, the variance of the additive noise in the projection data can be computed if $I(r,\theta)$ is known. In practice, a CT system measures $(I+i)$, therefore, for the purpose of computing the noise variance of the process $e(r,\theta)$, the measured photon intensity is assumed to be approximately equal to $I(r,\theta)$. Equation (3.1.11) suggests that the noise variance depends upon $I(r,\theta)$ and thus depends upon the attenuation provided by the object. In general, the noise process of projection data is thus non-stationary but the noise of the consecutive ray-sums within a particular projection is statistically independent.

3.1.1.2 Estimation of the Object Boundary from the Projection Data

The non-zero extent of the projection, at a particular angle, is determined by the boundary of the object in the same angular direction. This was used to estimate the approximate boundary of the object from the noisy projection data. Assume that the regular angular spacing among M projections is $\Delta\theta$ and the radial sampling distance for each projection is a . A binary (0 or 1) coefficient set $C(na, m\Delta\theta)$ was generated from the sampled projection data; if the projection data at the corresponding location was greater than three times the noise variance the coefficient was set to '1', otherwise it was set to '0'. The boundary mask of the object, $t(ja, ka)$, was then calculated as

$$t(ja, ka) = \prod_{m=0}^{M-1} c(na, m\Delta\theta) \quad (3.1.12)$$

$$n = \text{INT}(ja \cos(m\Delta\theta) + ka \sin(m\Delta\theta)) \quad (3.1.13)$$

Let a diagonal matrix T (N^2 by N^2) describe the boundary mask such that the N^2 diagonal elements are equal to the N^2 values of the boundary mask $t(ja, ka)$. The known boundary matrix and the noise vector can be used to modify equation (3.1.8) as follows [32].

$$P = KTF + E \quad (3.1.14)$$

This equation provides the discrete formulation of the image restoration problem from the projection data. In words: given the noisy projection data P , the beam profiles to describe the matrix K , the boundary mask T and an estimate of noise E , solve the system of equations (3.1.14) for the image vector F . In equation (3.1.7), the transformation of the original object into projection data is not the conventional image degradation model involving the convolution of two functions in 2-D Cartesian space. As a consequence of this, conventional image restoration approaches, such as inverse filtering or iterative deconvolution, are not applicable [53].

3.1.1.3 Constrained Least-squares Restoration

A MAP formulation of the CLS restoration from the projection data, based upon equation (2.3.21), results in the following system of equations

$$(R_{ff}^{-1}T + TK^TR_{ee}^{-1}KT)F' - R_{ff}^{-1}TF_m = TK^TR_{ee}^{-1}P \quad (3.1.15)$$

where R_{ff} and R_{ee} are the covariance matrices for the original image and the noise in the projection data, respectively, and F_m is the mean vector of F . The matrix R_{ee} is a diagonal matrix containing unequal diagonal elements since the noise process of projection data is spatially uncorrelated and non-stationary.

The size of the matrix $K^T R_{ee}^{-1} K$ is N^2 by N^2 for N by N pixels of the restored image. Therefore, in this direct form, it is very difficult to obtain a solution of these equations within reasonable computational and storage requirements. A simple procedure to avoid storing such a large matrix is to compute the multiplication $K^T R_{ee}^{-1} KTF$ in three steps. Firstly, the matrix multiplication KTF is the same as calculating the projections of a spatially bounded digital image using the finite size of the X-ray beam profile. Secondly, the elements of the vector KTF are divided by the position-dependent noise variance to give $R_{ee}^{-1} KTF$. In the third step, the multiplication of K^T with $R_{ee}^{-1} KTF$ is the same as backprojecting a set of sampled-projection data on a sampled image grid, however, the contribution to a particular pixel, from a backprojection at a particular angle, comes from a number of samples contained within the width of the beam profile. These three steps do not explicitly require the matrix K , instead, only the beam profile need be known. However, the procedures of computing the projections of a digital image using the beam profile and the weighted-backprojection require a large number of multiplications and additions. Therefore, an iterative solution of equation (3.1.15) would be very slow because of the complexity of computation per iteration. Further simplification in this process can only result from the assumption that the diagonal matrix R_{ee} must have identical diagonal terms which requires that the noise variance in the projection data is position-independent. The multiplications with the matrix $K^T R_{ee}^{-1} K$ then reduce to those with the matrix $K^T K$. The multiplications $K^T KTF$ can be performed by using the first and third steps of the procedure of multiplications $K^T R_{ee}^{-1} KTF$. However, an image resulting from weighted-backprojection of the projections of a single pixel element remains the same irrespective of the position of the pixel. Such an image can be considered as the spatially invariant PSF of the weighted projection-backprojection ($K^T K$) procedure. The multiplications $K^T KTF$ are then performed by using a 2-D discrete convolution between the spatially bounded image TF and this PSF. This approach was taken to form a solution of the system of equation (3.1.15). Multiplying both sides of this equation by R_{ee} , now uniformly diagonal, results in the following

$$(R_{ee} R_{ff}^{-1} T + T K^T K T) F' - R_{ee} R_{ff}^{-1} T F_m = T K^T P \quad (3.1.16)$$

If the random field corresponding to the original image is assumed to be stationary

then the second term on the left-hand-side of equation (3.1.16) can be ignored since it only changes the mean intensity in the restored image. If the pixel values in the original image are assumed to be spatially uncorrelated, then R_{ff} is a diagonal matrix. The matrix R_{ee} is also diagonal and one can assume that

$$\gamma I = R_{ee} R_{ff}^{-1} \quad (3.1.17)$$

where I is a unit-diagonal matrix and γ is ratio of the noise to the image powers.

The system of equations to be solved for obtaining a CLS solution is then given by

$$(\gamma T + T K^T K T) F' = T K^T P \quad (3.1.18)$$

This is the same as the regularized (minimum-norm) least-squares solution of equation (3.1.14). The parameter γ is added to diagonal terms of the symmetric matrix $T K^T K T$ in order to make it a positive definite matrix. If the boundary constraint is not used in the formulation of the problem then equation (3.1.18) is in the form of 2-D discrete convolution. In this case, a direct solution of equation (3.1.18) can be obtained using 2-D FFT. However, a boundary constraint has been shown to improve the images reconstructed from a fewer number of projections [32]. Therefore, in order to use the matrix T , an iterative scheme using the conjugate gradient method (Appendix III) is instead utilized [69]. In addition, the value of γ is chosen such that the solution vector F' satisfies the condition

$$\|P - K T F'\|^2 = \|E\|^2 \quad (3.1.19)$$

An initial value of γ can be calculated as the ratio of noise variance in the projection data and signal variance in the original image. However, signal variance in the original image is not known, *a priori*, and an initial value of γ is computed using the signal variance in the reconstructed image within the region inside the boundary. An initial solution of equation (3.1.18) is obtained for this value of γ . The value of γ is decreased by a small amount if the left-hand-side of equation (3.1.19) is larger than the right-hand-side and vice versa. The solution of equation (3.1.18) is then obtained using the modified γ , and this iteration over γ is terminated when the constraint (3.1.19) is satisfied. For a particular class of original images, however, the value of γ which satisfies the constraint (3.1.19) does not change significantly if the number of photons N_0 remains constant. Therefore, the double iteration in equations (3.1.18) and (3.1.19) can almost always be avoided for a particular class of original images.

3.1.2 Restoration from the Reconstructed CT Images

An image, $g(j, k)$, is reconstructed using the digital implementation of the filtered backprojection method from the noisy, measured projection data, $p_{sn}(na, m\Delta\theta)$.

The PSF of the CT system is computed by reconstructing an image from the projection data obtained for a point-object located at pixel (j, k) . Although, for finely sampled projection data, the PSF of this idealized CT system is independent of the position (j, k) , the non-bandlimited interpolation could result in a spatially variant PSF as indicated in section (2.2.2.2) [42].

Assume that a point-object is located at the origin of the (x, y) plane; the projections of this point-object are the same as the X-ray beam profile ($\Delta = a$ in equation (3.1.1)), sampled at regular intervals spaced a distance 'a' apart. The Fourier transform of the filtered projection is periodic (Figure 3.1.2a). Figure 3.1.2b shows the Fourier transform of the linear interpolating function [47]. Since an interpolation in the spatial domain can also be considered as a convolution between the filtered projection and the interpolating function, the Fourier transform of an interpolated projection (Figure 3.1.2c) is the product of the functions shown in Figures 3.1.2a and 3.1.2b. The interpolated projections are not band-limited because the linear interpolation allows some signal at frequencies beyond half the sampling frequency (π rads/mm).

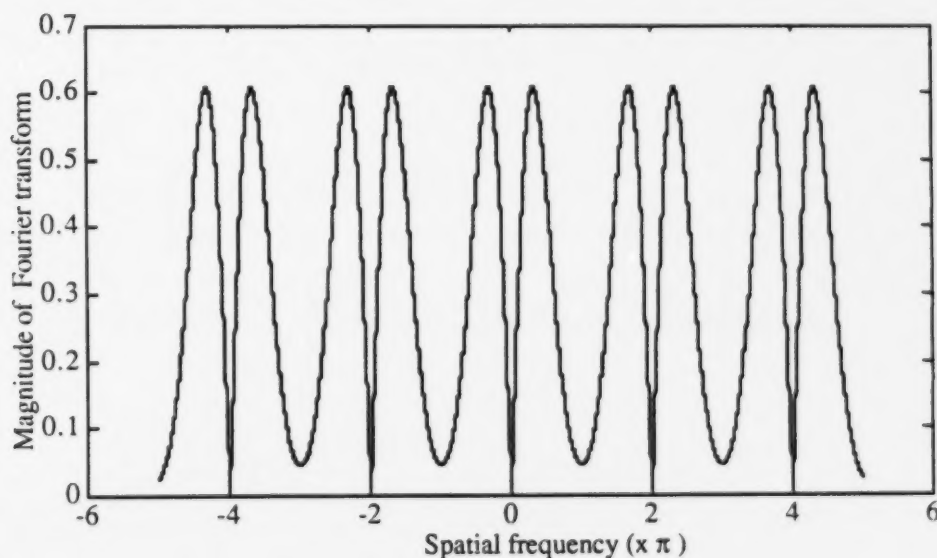


Figure 3.1.2a: Fourier transform of the filtered projection $q(na, m\Delta\theta)$ for a point-object computed by using the Gaussian X-ray beam profile.

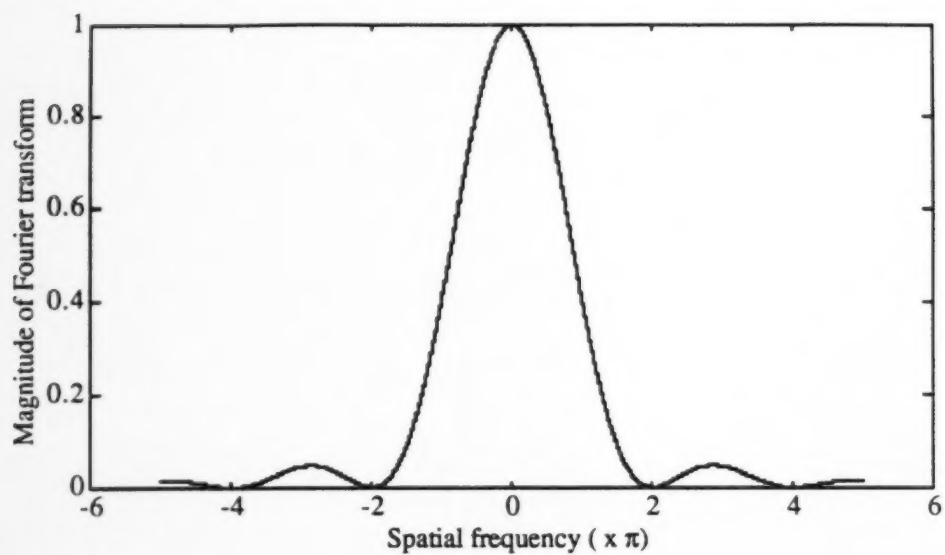


Figure 3.1.2b: Fourier transform of the linear interpolating function.

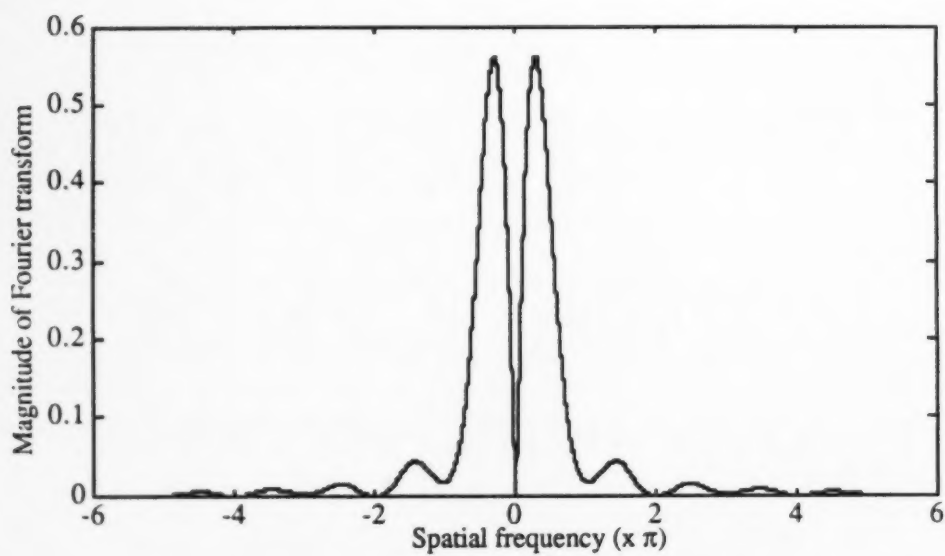


Figure 3.1.2c: Fourier transform of the filtered and interpolated projection.

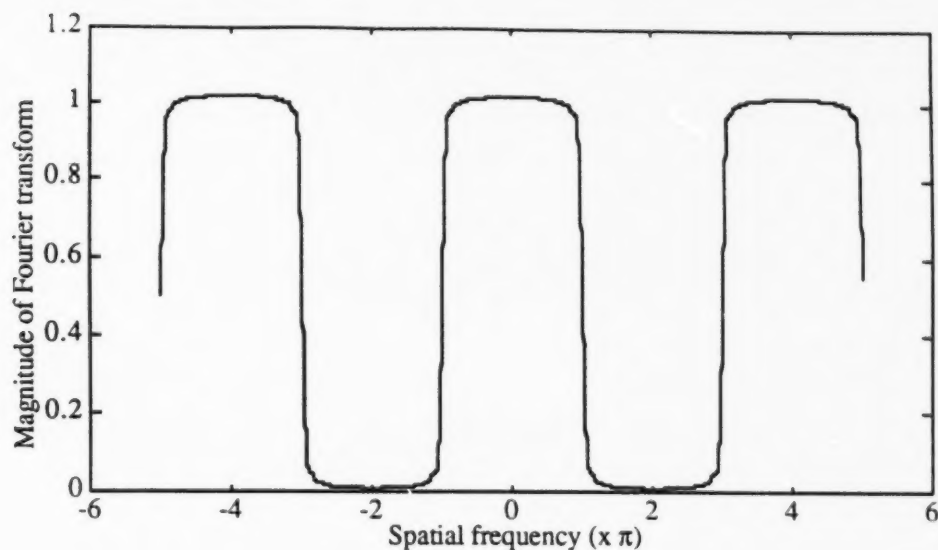


Figure 3.1.3a: Fourier transform of the interpolation filter used for doubling the size of filtered projection.

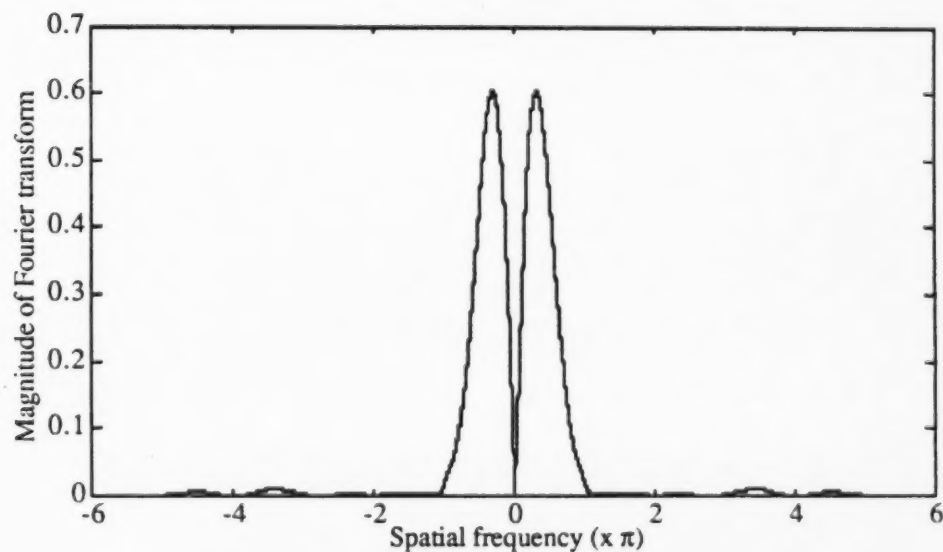


Figure 3.1.3b: Fourier transform of double-sized, linearly interpolated filtered projection (see text).

The magnitude of the signal beyond π rad/mm is determined by the sample spacing of projection at each angle. The projection data sampled using finer sample spacing results in a wider gap in the spatial frequency domain where the signal components are nearly zero. In

such a case, the filtered-interpolated projections become nearly band-limited. Such a fine sample spacing of the projection data, however, is not generally available with CT systems, and the linearly-interpolated projections are not band-limited. These filtered projections are backprojected on a sampled 2-D grid, and hence aliasing artifacts occur in the resulting CT images [45]. This particular problem can be solved by using an approximately band-limited form of interpolation.

An approximately band-limited interpolating filter is shown in the Fourier domain in Figure 3.1.3a. The filter was designed by applying a 39-sample-long triangular window to the ideal interpolating function in the spatial domain. Note that the Fourier transform of this filter is still periodic and that the period has doubled with respect to the Fourier transform of original filtered projection. The purpose of this filter is to double the size of the filtered projections, thereby halving the sampling distance. A filtered projection is interlaced with zeroes between the known samples and then convoluted with this filter function. A linear interpolation of the resulting filtered projection is then carried out. The Fourier transform of the interpolated filtered projection is shown in Figure 3.1.3b; the signal beyond π rad/mm is significantly reduced. In addition to reducing the signal leakage beyond π rad/mm, this interpolation method allows lesser attenuation for spatial frequencies below π rad/mm.

The interpolated projections (Figure 3.1.3b) of the point-object are then used to reconstruct the PSF of CT system. The interpolated projections resulting from this two-step interpolation are approximately band-limited and the PSF of a CT system becomes spatially invariant. Therefore, the discrete input-output image formation model can be written as a 2-D discrete convolution.

$$g(j,k) = \sum_{m=-M}^M \sum_{n=-M}^M f(j-m,k-n) h(m,n) + e(j,k) \quad (3.1.20)$$

where $f(j,k)$ is the original image and $e(j,k)$ is noise in the reconstructed image. The size of the PSF, $h(m,n)$, is $(2M+1)$ -by- $(2M+1)$ pixels in equation (3.1.20). In practice, 180 noiseless projections of a point-object were computed, and a 15-by-15 pixel area in the reconstructed image was taken as the PSF.

The noise $e(j,k)$ in the image is neither stationary, in either strict or wide sense, nor is it spatially uncorrelated. The noise in the image becomes spatially correlated due to the filtering, interpolation and backprojection of the projection data. The autocovariance function of this noise is spatially variant since the variance of noise in measured projections

depends on the transmitted intensities $I(r, \theta)$ at positions (r, θ) .

After extending the sampled images such that a circular convolution resulting from a 2-D DFT is the same as the linear convolution, the system of equations (3.1.20) can be written in matrix form. Let the vectors $\mathbf{G}$ and $\mathbf{E}$ represent a column-wise arrangement of the reconstructed image and noise respectively, and $\mathbf{H}$ be the block-circulant matrix obtained by similarly arranging the extended PSF. Then

$$\mathbf{G} = \mathbf{H}\mathbf{F} + \mathbf{E} \quad (3.1.21)$$

The restoration problem from the reconstructed images can be stated as follows: Given the blurred image, the PSF matrix and the properties of noise, determine the original image vector $\mathbf{F}$.

3.1.2.1 Wiener Filter Restoration

Let $G(k_1, k_2)$, $F(k_1, k_2)$ and $H(k_1, k_2)$ be DFTs of the reconstructed image, original image and the PSF respectively; k_1 and k_2 are the discrete spatial frequency indices in 2-D DFT space. Assuming that the pixels in the original image are statistically independent, the Wiener filter restoration, $F'(k_1, k_2)$, corresponding to equation (3.1.21) is given by

$$F'(k_1, k_2) = \frac{H^*(k_1, k_2)G(k_1, k_2)}{\gamma S_e(k_1, k_2) + |H(k_1, k_2)|^2} \quad (3.1.22)$$

where $S_e(k_1, k_2)$ is the NPS in CT images and $H^*(k_1, k_2)$ is the complex conjugate of $H(k_1, k_2)$.

The Wiener filter method is based upon the assumption that the noise in the reconstructed CT image is stationary. Therefore, it assumes that the noise in the projection data is independent of position (r, θ) . With this assumption, analysis of noise in the image can be carried out using the NPS [47, 67]. An analytical procedure of computing the NPS is given by Kijewski and Judy [47], a modified form of which appears in Appendix II. The modifications were made because of the two-step interpolation, and the effect of spatial truncation using a rectangular window of the ideal impulse response of the filter function used in the filtered backprojection method.

The parameter γ is adjusted to satisfy the following criterion

$$\|\mathbf{G} - \mathbf{H}\mathbf{F}'\|^2 = \|\mathbf{E}\|^2 \quad (3.1.23)$$

The right-hand-side of equation (3.1.23) is calculated as the sum of variances of $e(j,k)$ over the entire image. The position-dependent noise variance at each pixel is estimated from the noise variance in the projection data. A procedure for estimating the noise variance at each pixel in the image reconstructed using the filtered backprojection method has been given by Rosenfeld and Kak [70], and a modified form of this procedure appears in Appendix I. An initial estimate of the parameter γ was computed as the ratio of the noise to signal variance in the reconstructed image vector G . The noise variance and the signal variance were computed by using the pixels outside and inside the estimated object boundary, respectively, within the image vector G .

3.1.2.2 Locally Adaptive Restoration

The value of the parameter γ in equations (3.1.18) and (3.1.22) determines the degree of global smoothness or sharpness of the restored image. If this parameter is set to zero then equations (3.1.18) and (3.1.22) provide the pseudo-inverse solutions of the projection data and the reconstructed image formulations, respectively. As the value of this parameter is increased from zero, the resulting restoration gives different results from the corresponding inverse solutions to smoother images. In the derivation of equation (3.1.18) and the Wiener filter of equation (3.1.22), the Gaussian PDFs for the original image and the noise were assumed to describe stationary random processes. The noise in the projection data, and thus in the reconstructed images, is non-stationary. In addition, there are different regions in the original image which are either smoother or contain finer details. These regions would benefit from different values of γ since a single global value may result in excessive noise in the smooth parts of the image or excessive blur in those parts of the image containing finer detail [53, 57]. Therefore, a restoration technique based on local control of the regularizing parameter γ was sought. In the method developed, the facts that the finer image details are not uniformly distributed in the image, and that the noise in CT images, in a strict sense, is non-stationary, are utilized to locally regulate the degree of smoothing.

A restoration procedure which takes the spatially variant autocovariance function of noise in the reconstructed image into account would be very complex. Such a procedure would require the estimation of the spatially variant autocovariance function each time a different kind of object is scanned by a CT system. In the method developed, the autocovariance function of noise is, initially, assumed to be spatially invariant. The correlated noise is changed to white noise by filtering the reconstructed image using the normalized autocovariance function. Since the NPS is the Fourier transform of the

autocovariance function for the zero mean noise, this filter can be designed from the NPS calculated after assuming stationary noise in the projection data.

Let a block circulant matrix Φ describe a filter function, $\phi(m,n)$, which will whiten the spatially correlated noise in a CT image. The filter function $\phi(m,n)$ in the spatial domain can be calculating using inverse 2-D DFT (f^{-1}) of the square root of the reciprocal of NPS.

$$\phi(m,n) = f^{-1}\left(\sqrt{\frac{1}{S_e(k_1,k_2)}}\right) \quad (3.1.24)$$

After pre-whitening, the relationship between the original image and the filtered CT image can be written as

$$\Phi G = \Phi H F + \Phi E$$

or

$$G' = H' F + E' \quad (3.1.25)$$

In this equation, the vector E' represents an approximately spatially uncorrelated noise process. Therefore, the autocovariance function of the noise in the filtered image G' is very close to a δ -function. The noise in the filtered image G' can now be considered as non-stationary. Let a diagonal matrix D represent the noise variance at all pixel locations. The Gaussian PDF of the noise E' , which gives the conditional probability of observing the filtered image vector G' given one realization of the vector F from its class, can be written as

$$\text{Prob}(G'/F) = k_1 \exp[-(G' - H'F)^T D^{-1} (G' - H'F)] \quad (3.1.26)$$

The non-stationary image description can be incorporated by writing the *a priori* PDF as

$$\text{Prob}(F) = k_2 \exp(-\gamma F^T S F) \quad (3.1.27)$$

where k_1 and k_2 are the normalization constants. In equation (3.1.27), the *a priori* PDF is assumed to be Gaussian and pixel values in the original image are assumed to be statistically independent. It is also assumed that the mean pixel values do not change from one position to another, however, the variance of pixel values is position-dependent. Since the position-independent mean affects only the average intensity in the restored image

(equation 2.3.16), it is ignored in equation (3.1.27). The diagonal matrix S contains elements which depend inversely on the position-dependent variance of pixel values in the original image. The diagonal elements of S are normalized with respect to the parameter γ . Using Bayes theorem, the MAP restoration corresponding to equation (3.1.26) and (3.1.27) is the same as which is obtained by minimizing the functional (section 2.3)

$$J(F) = \gamma F^T S F + (H'F - G')^T D^{-1} (H'F - G') \quad (3.1.28)$$

The boundary constraint is applied indirectly through the matrix D^{-1} . Since the pixel elements outside the boundary are not required to be restored, the corresponding diagonal element for the matrix D^{-1} are set to zero. The j^{th} diagonal element of the matrix D^{-1} , corresponding to the j^{th} pixel inside the boundary of the object, is given by the reciprocal of the noise variance e_{jj}^2 at that pixel. The matrix S in this equation has non-zero elements only on diagonal given as

$$s_{jj} = \frac{1}{1 + \lambda \text{Max}(0, \sigma_{jj}^2 - e_{jj}^2)} \quad (3.1.29)$$

where σ_{jj}^2 is local variance, in a 5-by-5 pixel window, centered at the j^{th} pixel in the reconstructed image G , and λ is a tuning parameter. Ideally, the elements of matrix S are reciprocal of the variance of pixel values in the original image. Since this information is not available, the variances are approximately computed from the reconstructed image as in equation (3.1.29).

In equation (3.1.28) the larger the weight given to the first term ($F^T S F$), the smoother the resulting image. If the local variance is smaller than the noise variance, then the value of s_{jj} is equal to '1', alternatively if the local variance is larger than the noise variance, the value of s_{jj} is less than '1', allowing lesser smoothing for that pixel. Similarly, a smaller weight given to the second term of equation (3.1.29) would produce a smoother restoration which explains D^{-1} since a larger noise variance in an image region should lead to a smoother restored image in that region.

The conjugate gradient method is used again to solve the normal equations resulting from the minimization of equation (3.1.28), and the global parameter γ is adjusted to satisfy equation (3.1.23). The system of linear equations can again be determined by taking the derivative of equation (3.1.28) with respect to elements of vector F .

$$(\gamma S + H'^T D^{-1} H') F = H'^T D^{-1} G' \quad (3.1.30)$$

It should be noted that the matrices $(\gamma T + TK^T KT)$ and $(\gamma S + H'^T D^{-1} H')$ of equations (3.1.18) and (3.1.30), respectively, are both symmetric and positive definite. Therefore, the conjugate gradient method will converge to a solution.

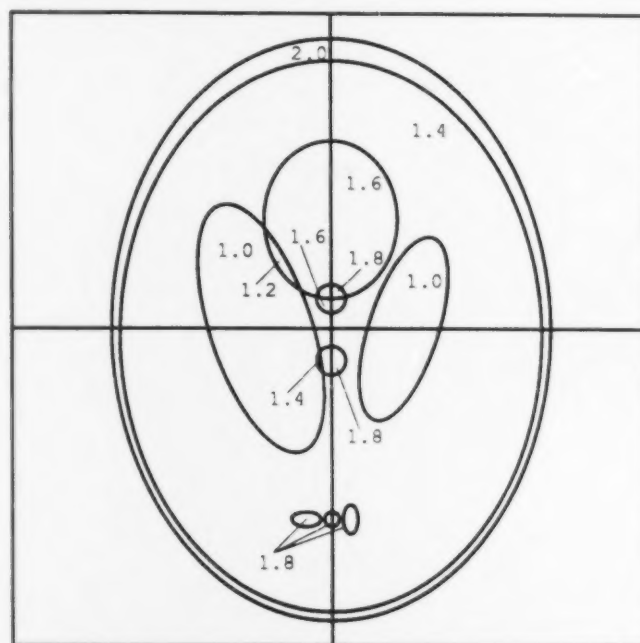


Figure 3.2.1: Geometry of Shepp and Logan head phantom. The numbers indicate the intensity and radius of the reconstruction circle (half the side of square) is 0.96 cm. For a list of parameters defining the diagram, see [12]

3.2 PERFORMANCE EVALUATION FOR TEST IMAGES

3.2.1 Test Images

Two types of test images were used to demonstrate the improvement obtained in the spatial resolution of CT images by using the restoration methods described. An image of the Shepp and Logan phantom [12] is shown in Figure 3.2.2a. To reduce the dynamic range, the attenuation values of the various ellipses were changed from those in [12] to those shown in Figure 3.2.1. A set of seven random pattern images was also generated for this study; one of these is shown in Figure 3.2.3a. In this image, the circular annulus is 0.5 mm wide (pixel size=0.3 mm) and intensity 2.0. The uniform background inside the annulus has intensity 1.0; within this background, there are randomly located pixels of intensity 2.0 (white pixels); the location (x,y) of these pixels was defined by using two uniformly distributed random numbers. The original images and reconstructed images are

of size 65-by-65 pixels. For the random pattern images, the white pixels in the original continuous object were assumed to be circles of diameter equal to the pixel size with their centers located at the pixel grid points.

3.2.2 Projection Data

Both the rms width of the Gaussian beam profile and the sampling distance of each projection was assumed to equal the pixel size. The line integrals $p_l(r, \theta)$ of the original continuous objects were calculated along finely spaced lines along r with an inter-line spacing 1/25th of the pixel size. It should be remembered that the calculation of line integrals requires summing of the attenuation properties of the continuous object along the particular lines which can be carried out by summing the individual products of the assumed intensities of the circles and ellipses in the object and the corresponding segment of the line contained within the circles and ellipses. The mean transmitted intensity, $I(r, \theta)$, integrated over the beam profile was then calculated as in equation (3.1.2) by approximating the integral using a trapezoidal rule. The observed transmitted intensity was calculated as the sum of $I(r, \theta)$ and a zero-mean, Gaussian random number of variance equal to $I(r, \theta)$. The measured projections were calculated as the logarithm of the ratio of N_0 and the observed transmitted intensity. In this manner, a total of 180 projections for each original object were calculated spaced at 1° over 180° . In order to add noise to the projection data, the total number of photons (N_0) was assumed to be 22000 in equation (3.1.1). The average signal to noise ratio was calculated from the true projection data based upon the mean transmitted intensity and the variance of the noise added to these projections. The mean signal to noise ratio in the projection data varied from 30 dB to 35 dB.

3.2.3 Reconstructed Images and Restoration Results

Reconstructed and restored images for the Shepp and Logan phantom and the random pattern object are shown in Figures 3.2.2 and 3.2.3, respectively. Images reconstructed using filtered backprojection with linear interpolation are shown in Figures 3.2.2b and 3.2.3b. For the two-step interpolation, applied before backprojection, the resulting images are shown in Figures 3.2.2c and 3.2.3c. Regularized least-squares restorations resulting directly from the projection data are shown in Figures 3.2.2d and 3.2.3d. Figures 3.2.2e and 3.2.2f show the results of applying the Wiener filter and the locally adaptive method, respectively, to the image shown in Figure 3.2.2c.

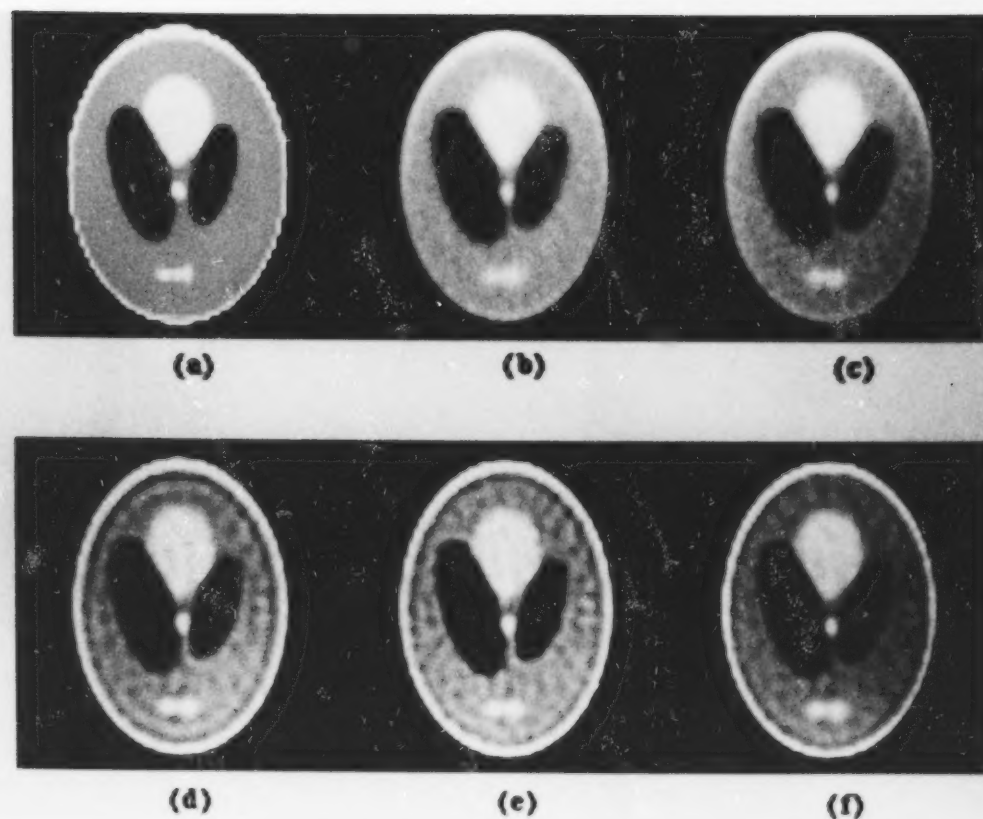


Figure 3.2.2: (a) Shepp and Logan phantom, sampled using 0.3 mm pixel size on a 65-by-65 grid, (b) image reconstructed using the filtered backprojection method and linear interpolation, (c) image reconstructed using the filtered backprojection method and two-step interpolation, (d) image restored from projection data, (e) restoration of part (c) using the Wiener filter, and (f) restoration of part (c) using the locally adaptive method.

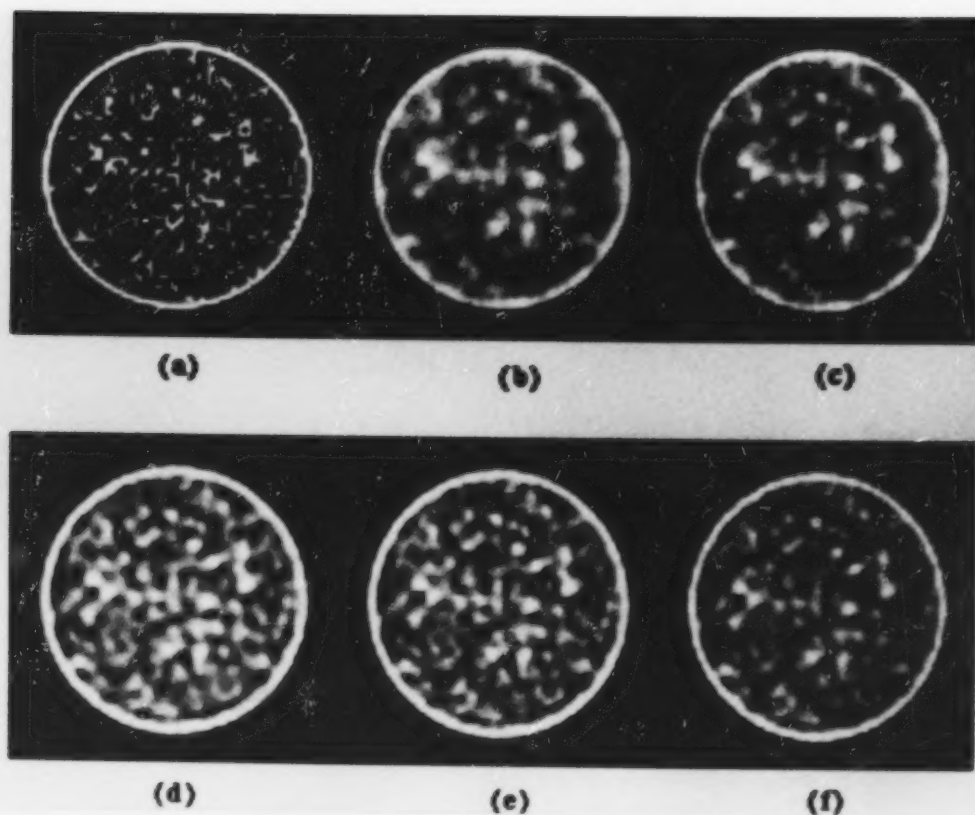


Figure 3.2.3: (a) Test image, the circular annulus is of thickness 0.05 cm with inner radius of 0.75 cm and intensity 2.0. Randomly located white pixels of intensity 2.0 are within the background of intensity 1.0, (b) image reconstructed using the filtered backprojection method and linear interpolation, (c) image reconstructed using the filtered backprojection method and two step interpolation, (d) image restored from projection data, (e) restoration of part (c) using the Wiener filter, and (f) restoration of part (c) using the locally adaptive method.

The results of applying the Wiener filter and the locally adaptive method to the image shown in Figure 3.2.3c are given, respectively, in Figures 3.2.3e and 3.2.3f.

The restored images of the Shepp and Logan phantom are easily evaluated based on sharpness of edges and smoothness of background. Restored images of the random pattern were evaluated with respect to two criteria: NSE and ROC. Considering first the NSE criterion, let F and F' denote the original test image and an image restored using one of the three methods. The %NSE was calculated as follows:

$$\%NSE = \frac{\|F - F'\|^2}{\|F\|^2} \times 100 \quad (3.2.1)$$

where the norms, $\| \cdot \|^2$, were computed over the non-zero region of the original image. Table 3.2.1 gives %NSE values for a set of seven reconstructed and restored images of the random pattern. The %NSE after the restoration is significantly lower than for both filtered backprojection reconstruction methods (Table 3.2.1). A slight reduction in the error results from the two-step interpolation in the filtered backprojection method. The simple Wiener filter also significantly reduces this error and improves spatial resolution. For the iterative methods, %NSE is slightly lower than that for the Wiener filter method. This is due to the fact that iterative methods are able to utilize other *a priori* information concerning the object during the iterative procedure to reduce the Gibbs oscillations [57]. In this regard, the locally adaptive method provides the lowest error since it uses the varying noise in the image to regulate smoothness. This is clearly indicated for the Shepp and Logan phantom (Figure 3.2.1) where the background area for the locally adaptive restoration is the smoothest of the three methods. It should be noted that all the *a priori* information used in these methods can be obtained from data collected by a CT scanner.

The second criterion (ROC) is based upon the ability of a hypothetical observer to classify white pixels and background. For reconstructed and restored images of the random pattern, histograms of pixel values corresponding to the background and white pixels inside the circular annulus in the original image set were generated (Figure 3.2.4). The histograms of the original image set have only two peaks separated by a value 1.0. The blurring resulting from the finite beam width in the projection process not only brings the two histograms closer together but also broadens them since the white pixels are randomly distributed. The correlated noise also broadens the histograms. Therefore, a restoration technique which removes the blur due to finite beam width, and does not introduce Gibbs oscillations and noise amplification is the best in the sense of the binary classification

system presented here.

As seen in Figures 3.2.4a and 3.2.4b, the two histograms overlap. From the histogram one can calculate the probability that a white pixel in the original image is classified as a white pixel in reconstructed image (true probability), as a function of the classification threshold. Similarly, the probability of falsely classifying a background pixel of the original image as a white pixel can also be computed as a function of the threshold. A plot of true probability against false probability for the same classification threshold gives

Image #	Filtered backprojection	Filtered backprojection with two step interpolation	Minimum norm from projections	Wiener Filter	Locally Adaptive Method
1	14.0	12.0	5.6	5.9	5.1
2	16.8	15.4	6.5	7.2	5.7
3	25.1	22.5	8.3	9.4	7.0
4	9.6	9.3	5.8	5.5	5.2
5	14.8	12.4	6.2	5.7	5.4
6	20.47	18.5	6.5	7.3	5.9
7	15.4	14.8	6.2	6.3	5.8

Table 3.2.1: Percentage normalized squared error (%NSE) between original and reconstructed images using different methods for the set of seven test images.

Reconstruction Method	Area under ROC curve
Filtered backprojection (Linear interpolation)	0.812
Filtered backprojection (two step interpolation)	0.825
Minimum norm from projections	0.861
Wiener filter	0.867
Locally adaptive restoration	0.870

Table 3.2.2 Area under receiver-operator-characteristics (ROC) curves resulting from the set of seven images using different reconstruction methods.

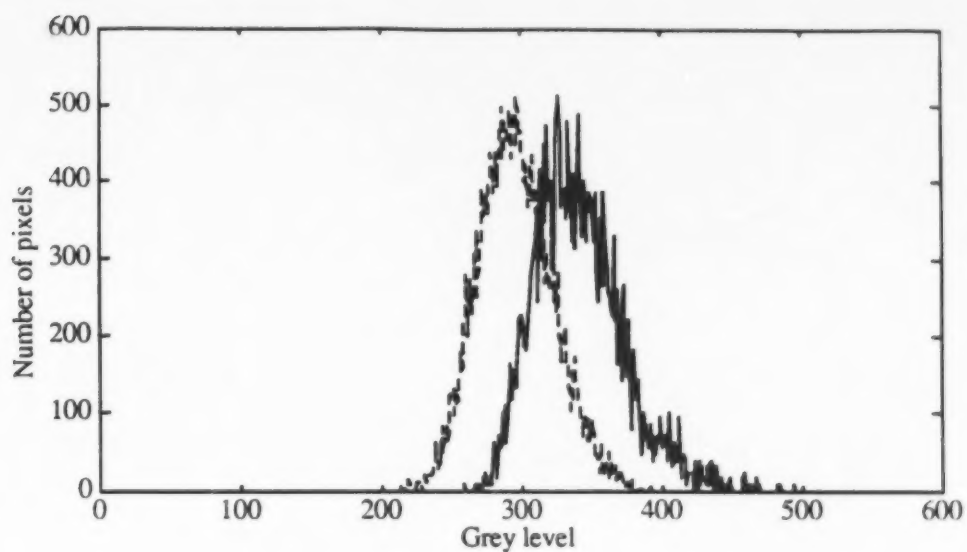


Figure 3.2.4a: Histograms of the background (dashed line) and white (solid line) pixels in the set of seven images restored using the locally adaptive method.

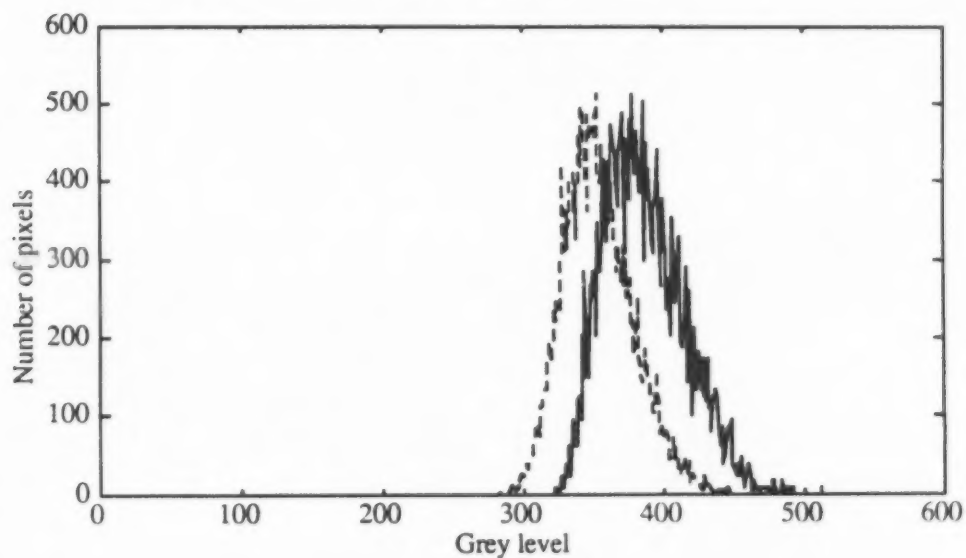


Figure 3.2.4b: Histograms of the background (dashed line) and white (solid line) pixels in the set of seven images reconstructed using filtered backprojection and linear interpolation.

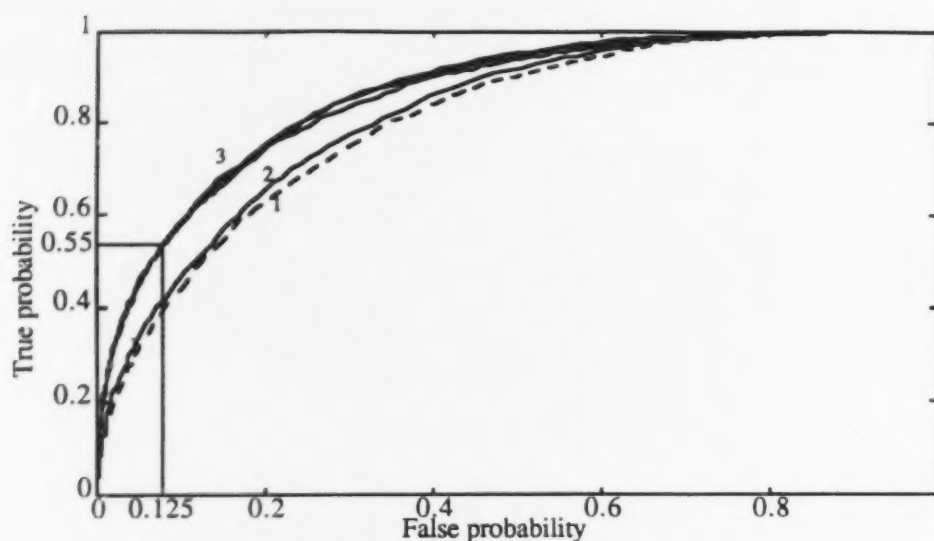


Figure 3.2.5: ROC characteristics for separating the white pixels from background in the set of seven images reconstructed using (lower solid line) filtered backprojection method with linear interpolation, (lower dashed line) filtered backprojection method with two-step interpolation, and (upper curves) restoration from projection data, Wiener filter and locally adaptive method.

the ROC curve [68]. For the original image the ROC curve would be such that the true probability is always 1.0, irrespective of the false probability. For the reconstructed images the ROC curves are shown in Figure 3.2.5. Note also that the area under the ROC curve for a perfect classification would be 1.0. The areas under the ROC curves resulting from the various methods are shown in Table 3.2.2.

Histograms for the background and white pixels in the entire image set of the random test pattern, for the filtered backprojection method with linear interpolation, and for the locally adaptive method are shown in Figure 3.2.4. The two peaks in the histogram for the locally adaptive method (Figure 3.2.4a) are more widely separated than those for the filtered backprojection method (Figure 3.2.4b), with little difference in line widths. The ROC curves for the image restoration methods indicate a 15% or greater chance of true classification than for the filtered backprojection method, at the same level for false probability (0.125). The area under the ROC curves is largest for the locally adaptive image restoration method (Table 3.2.2).

Some of the original images of random test pattern were also studied without adding noise to projections; the area under the ROC curves was increased from 0.812 to 0.84 for the filtered backprojection method with linear interpolation. The corresponding

improvement for the locally adaptive restoration method was from 0.87 to 0.94.

3.2.4 NSE vs. γ

In order to make a good choice of regularization parameter γ , the NSE can be analyzed as a function of γ . Considering the Wiener filter, the regularization using the NPS provides different results from those of an inverse filter. In the case of the Wiener filter, the difference between the restored and the original image at each spatial frequency can be written as

$$F(k_1, k_2) - F'(k_1, k_2) = \frac{\gamma S_e(k_1, k_2) F(k_1, k_2)}{|H(k_1, k_2)|^2 + \gamma S_e(k_1, k_2)} - \frac{H^*(k_1, k_2) E(k_1, k_2)}{|H(k_1, k_2)|^2 + \gamma S_e(k_1, k_2)} \quad (3.2.2)$$

where it is assumed that $E(k_1, k_2)$ is the 2-D DFT of the noise signal alone.

The %NSE can be analyzed using only the $\|F - F'\|^2$ term in (3.2.1) since the term $(100/\|F\|^2)$ is not affected by γ . The norm $\|F - F'\|^2$ is always less than or equal to the sum of the corresponding norms of two terms on the right-hand-side of equation (3.2.2). The squared-error can be analyzed by considering these two terms, one due to the regularization using the NPS $S_e(k_1, k_2)$ and the other due to the magnification of noise $E(k_1, k_2)$. As γ approaches to zero, the first term on the right-hand-side (regularization error) of equation (3.2.2) becomes zero, however, the second term becomes larger at high frequencies where $H(k_1, k_2)$ is very small. As the value of this parameter is increased from zero, the contribution of the first term to the squared-error increases while that of the second term decreases. The squared-error due only to the regularization is thus a monotonically increasing function of γ while the squared-error due only to the noise magnification is a monotonically decreasing function. Therefore, the squared-error first decreases as γ is increased from zero, reaches a minimum value and then increases. The %NSE for the Wiener filter as a function of γ is shown in Figure 3.2.6. It is clear that %NSE could only be decreased up to a certain minimum value by using the Wiener filter.

The locally adaptive method results in a linear but spatially variant restoration filter denoted by $d(j, k; m, n; \gamma)$. The difference between the original image $f(m, n)$ and the restored image $f'(m, n)$, in the spatial domain, can be written as

$$f(m, n) - f'(m, n) = \{f(m, n) - \sum_j \sum_k d(j, k; m, n; \gamma) f(j, k)\} + \sum_j \sum_k d(j, k; m, n; \gamma) e(j, k) \quad (3.2.3)$$

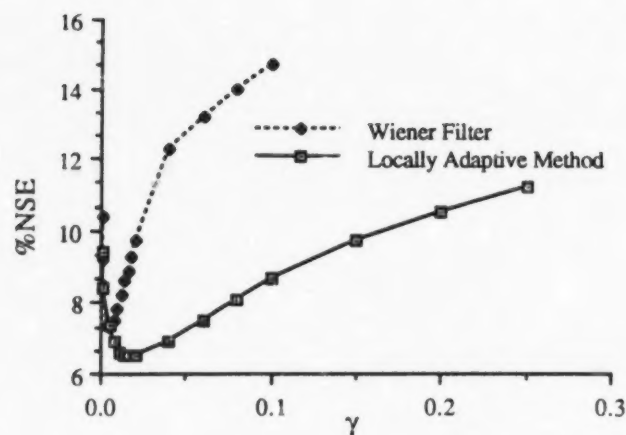


Figure 3.2.6: %NSE vs. γ curve for the Wiener filter and locally adaptive methods.

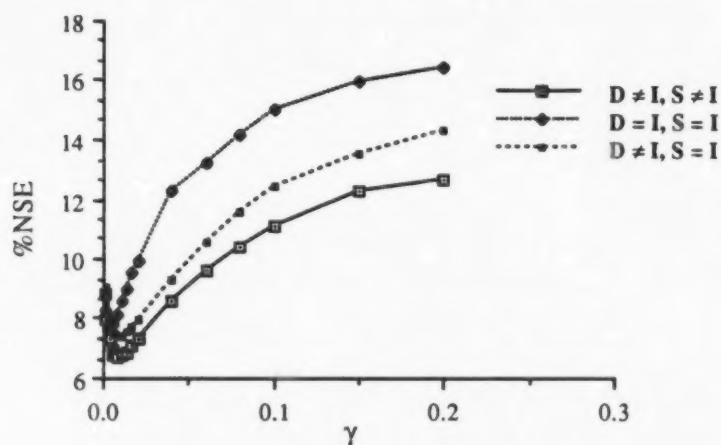


Figure 3.2.7: Effect of assuming the matrices D and S being unit diagonals on the %NSE vs. γ curve for locally adaptive method.

where d' is given by

$$d'(j,k; m,n; \gamma) = \sum_{j'} \sum_{k'} d(j',k'; m,n; \gamma) h(j-j', k-k') \quad (3.2.4)$$

The term $\|F - F'\|^2$ is again less than or equal to the corresponding norms of the two terms on the right-hand-side of equation (3.2.3). If the restoration filter is approximately equal to the inverse of the PSF then the function $d'(j,k; m,n; \gamma)$ is very close to a δ -function. However, due to the regularization performed using the pre-whitening step and matrices S and D , the restoration filter deviated from the inverse of the PSF. Therefore, the error resulting from the first term is known as the regularization error. An inverse to the PSF used for the restoration results in undue magnification of the noise in the second term of equation (3.2.3). Therefore, regularization is used to reduce the amplification of noise in the restored image. Adaptive restoration focuses on the design of the restoration filter $d(j,k; m,n; \gamma)$ such that the total error resulting from equation (3.2.3) is lower than obtained in equation (3.2.2). In the edge regions of the original image, the restoration filter is very close to the inverse filter such that the first term in equation (3.2.3) becomes very small. At the same time, if the image noise is very high in the edge region then the matrix D controls the undue magnification of the noise in the second term. In the smoother regions, the matrices S and D are adjusted to reduce the noise magnification error. As a result of this, the minimum %NSE obtained by the locally adaptive method is smaller than the minimum %NSE obtained using a Wiener filter. Increasing γ in the case of the locally adaptive method beyond the optimal range for the Wiener filter results into a smoother image, however, the smoother regions are smoothed more than the edge regions resulting in a lower %NSE for the locally adaptive method than the Wiener filter (Figure 3.2.6).

The locally adaptive method provides the similar %NSE vs. γ curve as in the Wiener filter if the matrices D and S are assumed to be unit-diagonals. The effect of individually setting these matrices to a unit-diagonal on the %NSE is illustrated in Figure 3.2.7.

3.3 CONCLUDING REMARKS AND SUMMARY

Before concluding this chapter, a number of remarks can be made about the formulation of the image restoration problem in CT. Although obtaining improvement in spatial resolution was a primary concern in this work, computational requirements were also a consideration. The Wiener filter method required the least number of computations since it is non-iterative.

For the iterative approaches, the locally adaptive method was faster than restoration directly from the projection data. This is because the matrix K^TK represents the PSF of a weighted projection-backprojection process which, in theory, is spatially unbounded. The

size of this PSF was the same as that for the blurred, reconstructed image (65-by-65 pixels). On the other hand, the matrix $\mathbf{H}$ represents the PSF of the projection-filtered-backprojection process. The major part of the energy of this PSF is limited to a 15-by-15 pixel neighborhood.

It should be noted that a Wiener filter method cannot always be used in practical CT scanners since the PSF is spatially variant. The purpose of using this method in the simulation was to demonstrate the usefulness of a simple approach to image restoration in CT when the PSF is approximated as spatially invariant.

Each of the restoration procedures described is based upon the assumption that pixel values in the original image are independent, which leads to a white signal power spectrum for the Wiener filter, and a diagonal autocorrelation matrix for the other two methods. This assumption is made since the autocorrelation function of the original image is not known, *a priori*. If the signal autocorrelation function is known, these procedures can then be easily modified by multiplying the autocorrelation matrix with the matrix $\mathbf{S}$ in equation (3.1.28), and the vector $\mathbf{TF}$ in the first term of equation (3.1.18). Similarly, the signal power spectrum can be used to divide the NPS in the Wiener filter case.

In summary, three simple approaches to image restoration have been evaluated; each method resulted in an improvement in spatial resolution. For noiseless projections almost perfect reproduction of the original image was obtained. Locally adaptive restoration of reconstructed CT images appears to be a very promising approach for noisy projection data, both in terms of the NSE and the area under the ROC curves.

4. ESTIMATION OF THE SPATIALLY VARIANT POINT SPREAD FUNCTION

In the previous chapter, image restoration techniques applied to the reconstructed CT images required the PSF and the NPS. The PSF of an ideal CT system was assumed to be spatially invariant in order to compare the results of image restoration methods applied directly to the projection data and to the reconstructed CT images. For the latter case, the position dependence of the PSF due to the non band-limited interpolation of the sampled filtered projections [42] was removed using an approximately band-limited interpolation. However, the PSF of a practical CT system is spatially variant despite adequate sampling of projection data and the modified form of interpolation used in the reconstruction algorithm. Due to the divergent (convergent) nature of X-ray beams for different detectors, the beam profiles are position dependent resulting in the spatially variant PSF. The position dependence of the PSF also results from the continuous rotational movement of the X-ray source and detector assembly of the scanner while collecting data. In order to apply an image restoration technique to the reconstructed CT images, the spatially variant PSF must be known at each pixel in the image. In this chapter, an approach to estimate the PSF of a practical CT scanner, based upon a model of the X-ray beam profiles, is described. The experimental measurement of the NPS will be discussed in the next chapter.

4.1 EXPERIMENTAL MEASUREMENT OF THE CT PSF

In addition to image restoration, measurement of the PSF is also required for the purpose of assessing the spatial resolution of a CT system, and comparing the performance of different scanners. The modulation transfer function (MTF), defined as the magnitude of the 2-D Fourier transform of the PSF, is used as an index of the spatial resolution of a CT scanner. Measurements of the MTF are usually made for commercial CT scanners to determine the spatial frequency bandwidth. Several different approaches have been used to obtain the MTF or the PSF of CT systems. In one method a thin, dense wire located at different positions in a water phantom is scanned [72, 73]. The resulting image of the wire is taken as the PSF of the CT system at that point. The main problem with this approach is that the wires cannot be exactly placed on the grid points of the sampled image, and thus the PSF is asymmetric in the spatial domain, and there is no general way of distinguishing whether the asymmetry is a characteristic of the PSF or is caused by the misregistration of the wire and the grid point [73].

If the 2-D PSF is assumed to be separable into a product of the two 1-D functions each dependent on one orthogonal direction in Cartesian coordinates, then an approach based upon the experimentally measured line spread function (LSF) can be used. The LSF is the integration of the 2-D PSF along the direction perpendicular to the line. The LSF can be measured by scanning a high-contrast thin wire placed along each of the orthogonal directions in the scan plane in a test phantom. However, this approach provides insufficient number of data points for describing the LSF [74]. An experimental measurement of the edge response function (ERF) is, therefore, made and the LSF is calculated by taking the derivative of the measured ERF in the direction perpendicular to the edge [74, 75]. The LSF in the other orthogonal variable can be estimated by measuring the ERF of the same edge rotated by 90° . Since the PSF in two dimensions cannot be computed without the assumption of separability, the usefulness of this method is limited as far as the image restoration is concerned. In addition, this method is generally used to obtain the PSF only at one point located at the center of the scan field.

The MTF of a CT system can be determined by scanning a cyclic bar pattern and measuring the amplitude of the resulting pattern in the image. The number of bars per unit distance can be varied to provide the MTF magnitude at discrete spatial frequencies [76]. However, the spatially variant PSF in the spatial domain cannot be determined using this procedure. Moreover, noise in the projection data is propagated to the reconstructed images limiting the accuracy of both the LSF and MTF methods.

In a CT system where resolution is spatially variant, PSF measurements are required at a number of points. Therefore, a test phantom containing a large number of wires placed at different locations is generally scanned to measure the PSF simultaneously at several points. To this end parametrization of the measured PSF [77] by fitting 2-D composite functions is a useful technique since spatial variations in the PSF are mapped into spatial variations in the parameters. However, parametrization of the PSFs measured from images of thin, dense wires suffers from the problems of misregistration and noise. Also, fitting 2-D composite and non-linear functions to a relatively small set of noisy pixel values is not reliable. For a commercial system which uses a large filament current for the X-ray tube [77], this method has been shown to work well since the noise in the images is significantly lower than in the images obtained by special purpose, low dose CT systems.

Due to these limitations on the techniques for experimentally measuring the PSF of a CT system, an approach based on modeling the X-ray beam was used. In the following sections, the basis for computing the PSF from the known beam profiles is provided. The

X-ray beam profiles, in turn, are obtained from the measured focal spot intensity and detector sensitivity distributions.

4.2 CALCULATION OF THE PSF USING THE X-RAY BEAM PROFILES

Let a real valued function $f(x,y)$ represent the attenuation properties of an object in the x - y plane. In Figure 4.2.1, the finite sizes of the X-ray focal spot and detector collimator are denoted by S and D respectively. Let $K_0(x,y; r,\theta)$ represent the X-ray beam profile at an angle θ and a distance r from the origin. The function $K_0(x,y; r,\theta)$ defines the number of photons passing through a point (x,y) which are emitted from all the points along the focal spot and detected at any point along the detector. The narrow strip in Figure 4.2.1 shows the region over which an X-ray beam profile is defined. In the previous chapter, the beam profile $K_0(r)$ was independent of x , y , and θ due to an assumption that only the parallel rays form the beam. In a practical CT system, a point on the detector D is capable of detecting photons which are emitted from a large number of points on the focal spot S .

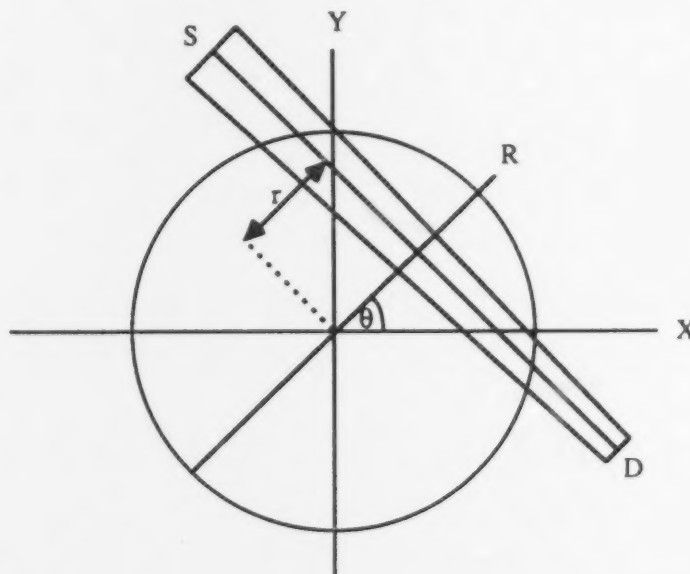


Figure 4.2.1: Data collection geometry for a CT scanner. The measured projection is an average of the 2-D function over the finite size X-ray beam (narrow strip).

Similarly, the photons emitted from a single point on the focal spot may travel along a number of lines, with different orientations with respect to the central ray, thereby,

reaching the detector at more than one point. As a result, the X-ray beam is formed by a bundle of convergent rays and the total number of photons in a converging beam becomes a function of the distance from the focal spot. Since the distance of a point from the focal spot is dependent upon the position (x,y) , the X-ray beam profile is denoted as a function of the x and y variables. Moreover, a fan-beam CT scanner utilizes different detectors for obtaining the data at different values of r and θ so that, the beam profile $K_0(x,y; r,\theta)$ is also a function of (r,θ) .

Under the assumption that the line integrals of the function $f(x,y)$ do not change significantly over the width of the X-ray beam profile, a linear relationship between the projections $p_s(r,\theta)$, measured due to the finite size of the X-ray beam, and the function $f(x,y)$ can be derived using an approach similar to the previous chapter (section 3.1.1). That is,

$$p_s(r,\theta) = \int \int f(x,y) K_0(x,y; r,\theta) dx dy \quad (4.2.1)$$

In this weighted-superposition integral of the function $f(x,y)$, the total number of photons contained in the X-ray beam has been normalized to unity.

The PSF of a CT system is the same as the image reconstructed from the measured projections of a point-object. A point-object of unity linear attenuation coefficient located at (x_0,y_0) results in a ray-sum of magnitude $K_0(x_0,y_0; r,\theta)$ when the X-ray beam is positioned at (r,θ) . Therefore, the PSF, $h(x,y; x_0,y_0)$, due to this point can be written as

$$h(x,y; x_0,y_0) = \int [K_0(x_0,y_0; r,\theta) * \rho(r)]_{r=r_0} d\theta \quad (4.2.2)$$

where $\rho(r)$ is the filter function used in a filtered backprojection method and $r_0 = x \cos\theta + y \sin\theta$.

Equation (4.2.2) describes the filtered backprojection method of image reconstruction from projections similar to the equations (2.1.1, 2.1.10 and 2.1.11) in chapter 2. This equation is written in the integral form in order to highlight the relationship between the PSF and the X-ray beam profile. In practice, the sampled projections of a point-object, computed using the known X-ray beam profiles, are used in a discrete implementation of the filtered backprojection algorithm to form a PSF image. An

approximately band-limited, two-step interpolation (section 3.1.2) of the filtered projections is used before backprojection in order to avoid the 2-D aliasing [45] which results when only linear interpolation is used for the backprojection process. It can be seen that the PSF due to any point-object located at an arbitrary point (x_0, y_0) can be computed if the X-ray beam profiles for all the values of $(x_0, y_0; r, \theta)$ are known.

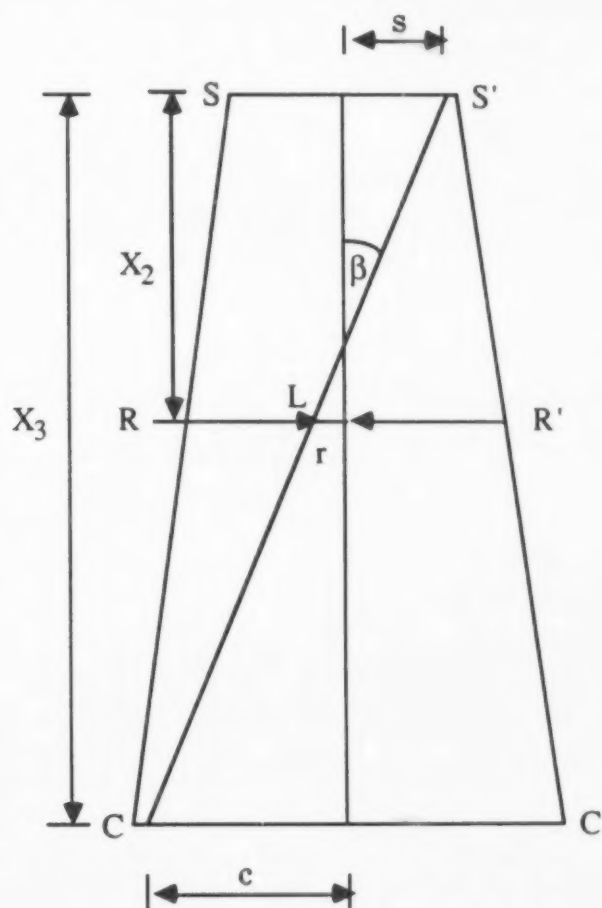


Figure 4.2.2: Geometry of focal spot (SS'), detector collimator aperture (CC'), and the slice through the beam profile (RR').

4.2.1 Modeling of the X-ray Beam profile

Although, the beam profile at a fixed position (r, θ) is a 2-D function of (x, y) , the following description is used to model the beam profile in terms of 1-D cross-sections

located at any arbitrary perpendicular distance from the focal spot. Considering the geometrical arrangement in Figure 4.2.2, the focal spot and detector collimator are drawn as SS' and CC' respectively. The development of the model is considered for a cross-section along RR'. Let the distributions of the photon intensity emitted from the focal spot SS' (Figure 4.2.2) and of the detector sensitivity along the detector collimator CC' be denoted by N(s) and W(c) respectively. The X-ray beam profile is defined in the region enclosed by the trapezoid SS'C'C. Let B₀(r) denote a 1-D slice through the X-ray beam profile at a distance X₂ from the focal spot. The distribution B₀(r) is defined as the total number of photons which pass through point L, at a distance r along RR' from the vertical line connecting the mid points of the focal spot and detector aperture, and are detected (Figure 4.2.2). Therefore

$$B_0(r) = \int_{-\pi}^{\pi} N(s) W(c) d\beta \quad (4.2.3)$$

where angle β represents a ray which intersects the lines CC', RR' and SS' at distances c, r and s, respectively, from the vertical.

Since the widths of the focal spot and detector collimator aperture are very small compared to the distances between the focal spot and the detector collimator, the angle β is always very small so that

$$\beta \approx \frac{c-r}{X_3-X_2} \quad (4.2.4)$$

$$d\beta \approx \frac{dc}{X_3-X_2} \quad (4.2.5)$$

where equation (4.2.5) results when all the rays crossing the slice RR', at a fixed distance r from the vertical, are considered. In addition, s is related to c as follows

$$s = \frac{X_2 c}{X_3 - X_2} - \frac{X_3 r}{X_3 - X_2} \quad (4.2.6)$$

Substituting equations (4.2.4, 4.2.5, 4.2.6) into equation (4.2.3) results in

$$B_0(r) = \int_{-\infty}^{\infty} N\left(\frac{X_2 c}{X_3 - X_2} - \frac{X_3 r}{X_3 - X_2}\right) W(c) \frac{dc}{X_3 - X_2} \quad (4.2.7)$$

Setting $v = (X_2/X_3)c$ and $dv = (X_2/X_3)dc$, this expression changes to

$$B_0(r) = \int_{-\infty}^{\infty} N\left[\left(\frac{X_3}{X_3-X_2}\right)(v-r)\right] W\left[\frac{X_3 v}{X_2}\right] \left[\frac{X_3}{X_2(X_3-X_2)}\right] dv \quad (4.2.8)$$

If the functions N and W are assumed to be real and even, equation (4.2.8) represents a convolution expressed as the following

$$B_0(r) = \frac{X_3}{X_2(X_3-X_2)} N\left(\frac{X_3 r}{X_3-X_2}\right) * W\left(\frac{X_3 r}{X_2}\right) \quad (4.2.9)$$

or

$$B_0(r) = \frac{M}{X_2} N(Mr) * W\left(\frac{M}{M-1} r\right) \quad (4.2.10)$$

The magnification factor M is given by

$$M = \frac{X_3}{X_3 - X_2} \quad (4.2.11)$$

and $*$ is a 1-D convolution operator, and the functions $W(c)$ and $N(s)$ are re-scaled along the slice RR' such that $s=Mr$, and $c=(M/M-1)r$, respectively.

The X-ray beam profile along a line at a particular distance from the focal spot, is thus given by convolution of the appropriately scaled focal spot intensity and detector sensitivity distributions. If these functions are known, then the X-ray beam profiles can be completely described. This method for modeling X-ray beam profiles is similar to that used for determining sensitivity and exposure profiles along the slice thickness [78]. It should also be noted that the direction along which the beam profile is defined by this model is perpendicular to the central ray of the beam. Therefore, the 2-D X-ray profiles can be completely determined in terms of the 1-D slices at all distances from the focal spot.

4.2.2 Parameter Estimation for Fan-beam Scanner

The geometry of one type of fan-beam scanner is shown in Figure 4.2.3. The detectors are arranged in an arc at a fixed distance from the focal point (S). In an X-ray tube, the emitted photons are attenuated within the anode material on the near side of the anode heel. Therefore, in order to obtain a nearly symmetrical photon distribution within the fan-beam, the anode is tilted with respect to the perpendicular to the tube axis. Decreasing this tilt angle allows one to use greater area of the target on the anode for the same size of the effective focal spot but reduces the useful angle of the fan-beam. Due to this slant of the anode, there is always a detector which is aligned nearly parallel to the

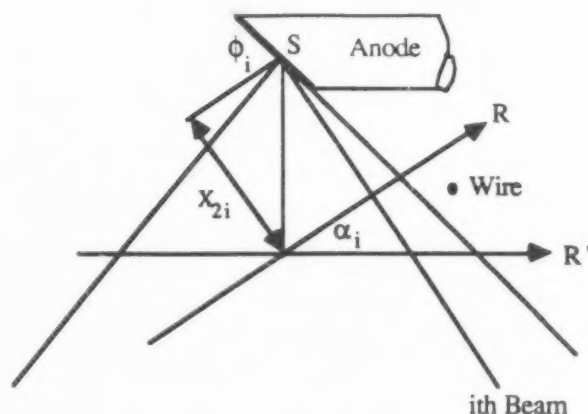


Figure 4.2.3: Geometry of a fan-beam CT system.

anode surface.

In practice, an X-ray tube (pre-patient) collimator is used to limit the photon distribution into a fan-beam on the far side of the anode heel. Such a collimator was not included in the development of the X-ray beam profiles model since this collimator is much wider than both the focal spot and the detector (post-patient) collimator.

X-ray beam profiles for each of the detectors in the fan-beam can be computed using the model proposed in previous section. Since the detectors are located at different angular orientations with respect to the anode, the focal spot intensity distribution seen by each detector is different. If N_{anode} is the focal spot intensity distribution along the anode then the effective focal spot intensity distribution seen by the i^{th} detector $N_i(r)$ is given by

$$N_i(r) = N_{\text{anode}}\left(\frac{r}{\cos \phi_i}\right) \quad (4.2.12)$$

$N_i(r)$ is defined along a direction perpendicular to the central ray for the i^{th} detector. The focal spot intensity distribution seen by the i^{th} detector at a distance r from its central ray is thus the same as the focal spot intensity distribution along the anode at a distance of $(r/\cos \phi_i)$ from the focal point. The effective focal spot intensity distributions for detectors on the left of the i^{th} detector are therefore wider than those on the right.

4.2.2.1 Measurement of Focal Spot Intensity Distribution

Measurements of focal spot intensity distribution were made using a special purpose fan-beam CT scanner [79] developed for appendicular bone density

measurements. This scanner has 26 detectors spaced 2° apart in a 50° fan-beam. The anode heel angle of the X-ray tube is 25° . In this particular device, in addition to a 360° rotation, the fan-beam can be linearly translated both in the horizontal and vertical directions.

A 0.23 mm diameter steel wire was placed outside the fan-beam as indicated in Figure 4.2.3. Attenuation profiles for this wire were measured for detectors 5-26 (left to right) as the fan-beam was translated horizontally. Detectors 1 to 4 were not included due to the limited horizontal translation available. The detectors measure the transmitted intensity as the source-detector assembly translates across the wire. The attenuation profile of the wire is then computed as the logarithm of the ratio of the incident to transmitted intensity as discussed in the previous chapter. For this CT system, the accumulated photon counts for each detector can be sampled at any arbitrary spatial intervals provided the counts per interval remain significantly high. Therefore, the spatial sampling interval was taken to be 2.226×10^{-3} cm and the translation speed was accordingly reduced to measure the attenuation profiles.

For the i^{th} detector (Figure 4.2.3), the measured attenuation profile depends upon the distribution of photons in the beam along the direction of movement R' , at a distance X_{2i} from the focal spot. This 1-D distribution is given by a slice through the X-ray beam profile along the direction R' . However, the proposed model can only be used to define slices along directions (R) perpendicular to central ray of the beam. Slices through the X-ray beam profiles in the directions R' and R are approximately related: let $B_{0i}(r)$ and $B'_{0i}(r')$ denote slices through the X-ray beam profile of the i^{th} detector at a distance X_{2i} from the focal spot, along the directions R and R' respectively, then

$$B'_{0i}(r') = B_{0i}\left(\frac{r'}{\cos \alpha_i}\right) \quad (4.2.13)$$

where r' is the distance along the direction of movement of the fan-beam and α_i is the angle between slices B'_{0i} and B_{0i} .

In order to take the width of the steel wire into account, let $p_i(r')$ represent the profile of line integrals of this wire. Assuming that the cross-section of the wire is a circle of radius Ω , the function $p_i(r')$ is expressed as

$$p_i(r') = 2\sqrt{\Omega^2 - r'^2} \quad (4.2.14)$$

where it is assumed that the linear attenuation coefficient of the material of the wire is unity,

and the center of the wire is located at $r'=0$.

Using equation (3.1.6), the measured attenuation profile, $p_{si}(r')$, for the i^{th} detector is given by the convolution of $p_l(r')$ and $B'_{0i}(r')$.

$$p_{si}(r') = p_l(r') * B'_{0i}(r') \quad (4.2.15)$$

Using equations (4.2.10) and (4.2.12), $p_{si}(r')$ can be written in terms of the focal spot intensity and detector sensitivity distributions,

$$p_{si}(r') = \frac{M_i}{X_{2i}} N_i \left[\frac{M_i r'}{\cos \alpha_i} \right] * W \left[\frac{M_i r'}{(M_i - 1) \cos \alpha_i} \right] * p_l(r') \quad (4.2.16)$$

Therefore, measured attenuation profiles for each detector can be related to the focal spot intensity and detector sensitivity distributions. Due to collimator design, detector sensitivity distribution along all detector collimators is the same. Since the effective focal spot intensity distribution for a detector aligned along the anode is a delta function, the measured attenuation profile for the rightmost detector shown in Figure 4.2.3 is thus defined as

$$p_{si}(r') = \frac{M_i}{X_{2i}} W \left[\frac{M_i r'}{(M_i - 1) \cos \alpha_i} \right] * p_l(r') \quad (4.2.17)$$

The function N_i can be calculated using a deconvolution of the measured attenuation profile for the i^{th} detector with respect to that for the right most detector. Also differences in magnification factor M_i and angle α_i must be removed in order to express the resulting effective focal spot intensity distribution as a function of the variable r' . Since deconvolution of noisy data may cause problems, an alternative approach was taken.

Let the sampled attenuation profile for each detector be denoted by $P_{si}(k)$. Assuming these functions are symmetric, the peak position, P_i , from each of these sampled data was determined as the first moment using the following

$$P_i = \frac{\sum_k k P_{si}(k)}{\sum_k P_{si}(k)} \quad (4.2.18)$$

The summations in the numerator and the denominator of equation (4.2.18) were

carried out for 150 points centered at the maximum value in $P_{si}(k)$. A 151 element segment of $P_{si}(k)$, centered at the peak position, was then extracted and the following Gaussian function was fitted using least-squares.

$$G_i(r_i') = A_i \exp\left(-\frac{(r_i' - b_i)^2}{2\Delta_i^2}\right) \quad (4.2.19)$$

where A_i and Δ_i are the amplitude and root-mean-squared (rms) width respectively for the Gaussian function, and b_i indicates that the origin of the continuous attenuation profile may be situated at a sub-sample location. Since the measured attenuation profile is calculated as a convolution of three different functions, a Gaussian form is a reasonable assumption.

In order to take the measurement noise into account, the least-squares fit was replaced with a χ^2 fit where

$$\chi^2 = \sum_{k=0}^{150} \frac{(P_{si}(k) - G_i(r_i'))^2}{\sigma_i^2(k)} \quad (4.2.20)$$

and r_i' is computed at the sampled positions of the measured data spaced at 2.226×10^{-3} cm. Assuming that the noise in the measured data is uncorrelated, the noise-variance $\sigma_i^2(k)$ at each sample-point can be computed using the measured photon intensity as described in chapter 3 (section 3.1.1.1).

An iterative approach for minimizing the χ^2 was carried out using the *Levenberg-Marquardt method* [80]. The χ^2 is a non-quadratic function of the parameters of the model, and the system of equations resulting from taking the derivative of χ^2 with respect to the parameters is non-linear. Therefore, the χ^2 -fit is a non-linear optimization problem. The proposed algorithm minimizes the χ^2 by using a steepest descent search in the parameter space. In a general non-linear optimization approach, the modified parameter vector in an iterative procedure is obtained by determining the minima along the search direction given by the steepest descent. The minima along the steepest descent direction is obtained by carrying out a binary search since the matrix of the second order derivatives (the Hessian matrix) of the non-linear function with respect to the model parameters is generally not available. In a non-linear χ^2 fit, however, the Hessian matrix is always available since the explicit model is a continuous function of the parameters. In the *Levenberg-Marquardt method*, the incremental improvements in the parameters are thus obtained by solving a linear system of equations which relates the parameter improvement

vector to the current steepest descent direction vector via the Hessian matrix. When fitting the Gaussian model to the attenuation profiles, this iterative procedure was allowed to run till χ^2 was reduced to the number of degrees of freedom, i.e. 148. A typical graph of fitted result is given in Figure 4.2.4 showing the measured data and the fitted Gaussian beam profiles overlaid. The fitted Gaussian function to the beam profile is very close to the measured data except at some points due to the measurement noise. Therefore, the rms width and the amplitude parameters for the beam profile for detectors 5 to 26 were estimated in this manner.

As a result of parametrization, both the deconvolution and the scaling differences in equations (4.2.16) and (4.2.17) can be performed in a simple manner. The deconvolution of two Gaussian functions involves subtracting the squared rms widths and dividing the amplitude parameters.

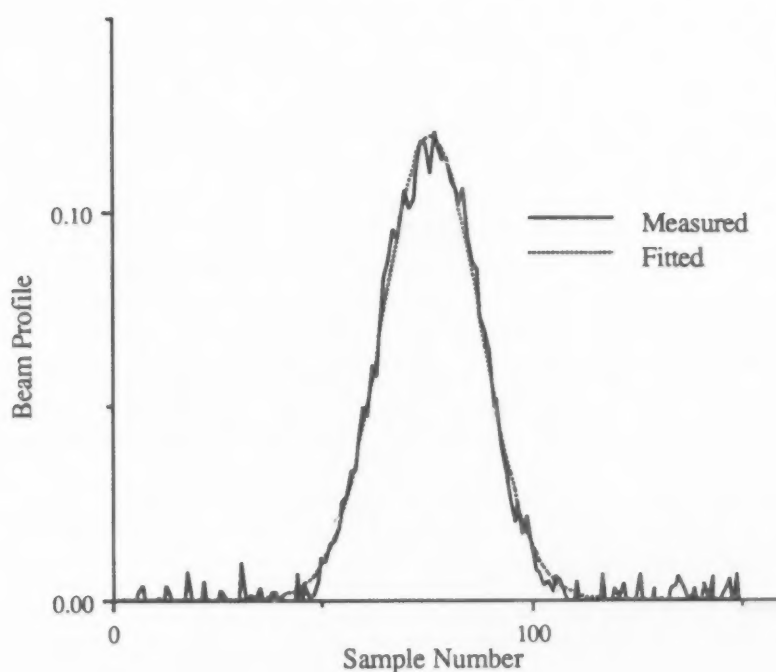


Figure 4.2.4: The measured (solid) and fitted Gaussian (dotted) beam profiles for detector 14.

Similarly, scaling in the independent variable can be achieved by properly scaling the rms widths, Δ_i . The scaling factor due to the angle α_i can always be computed if the angular orientations of the detectors with respect the line vertical to the direction R' (Figure 4.2.3), and the anode heel angle are known. As a result of the deconvolution of the two Gaussian functions, the functions N_i are also Gaussian for which the rms widths can be computed using the above procedure. The rms width of the focal spot intensity distribution along the anode can be computed by applying the scaling of equation (4.2.12) to the rms width for each of the N_i . The resulting rms widths for N_{anode} are given for 21 detectors (i. e. excluding the detector parallel to the heel) in Table 4.2.1.

The X-ray beam profile for a practical CT scanner is, of course, a 3-D distribution. In the direction perpendicular to the slice plane, the behavior of the beam profile is expected to be similar [78]. However, if the steel wire is assumed to be accurately aligned along the perpendicular to the slice plane, the variations in this direction do not alter the discussion concerning variations within the slice plane.

4.2.2.2 Attenuation Profiles Predicted by the Model

The model [equation (4.2.16)] can be used to predict attenuation profiles for a thin steel wire at any distance from the focal spot. A second experimental study was conducted to verify the prediction of the model. Attenuation profiles for detectors 5 to 26 were measured for a number of positions of the steel wire. For a particular detector, the attenuation profile depends on the distance from the focal spot since the magnification factor changes. Profiles were parametrized as 1-D Gaussian functions and their corresponding rms widths were calculated as a function of distance from the focal spot.

Using equation (4.2.16), the attenuation profile for each detector was calculated using a focal spot intensity distribution described as a Gaussian function with an rms width taken as the mean of the 21 measurements shown in Table 4.2.1. These calculations were performed for the same distances from the focal spot as in the experimental measurements. The sensitivity distribution of each detector was measured by placing the steel wire very close to the collimator; sensitivity was found to be approximately uniform within the slice plane over the width of collimator. Attenuation profiles calculated in this manner were again modeled as 1-D Gaussian functions and the rms widths calculated.

For an accurate model the rms widths of both the measured and the calculated data must vary identically as a function of distance from the focal spot.

Beam Number	Angle (Degree)	Predicted focal spot rms width (mm)	% SEE
1	42	0.735	0.54
2	40	0.745	0.54
3	38	0.737	0.24
4	36	0.742	0.29
5	34	0.744	0.19
6	32	0.742	0.28
7	30	0.748	0.50
8	28	0.740	0.35
9	26	0.751	0.41
10	24	0.742	0.52
11	22	0.753	0.51
12	20	0.747	0.50
13	18	0.754	0.41
14	16	0.754	0.52
15	14	0.746	0.90
16	12	0.753	1.19
17	10	0.754	1.39
18	8	0.756	0.64
19	6	0.773	1.17
20	4	0.743	1.21
21	2	0.807	0.99
22	0	Rightmost beam	0.98

Table 4.2.1: Comparison of rms widths for the Gaussian focal spot intensity distribution (column 3) predicted by each detector and %SEE (column 4) for regression of measured on predicted spread parameters of the attenuation profiles. The second column gives the orientation of the detectors with respect to anode.

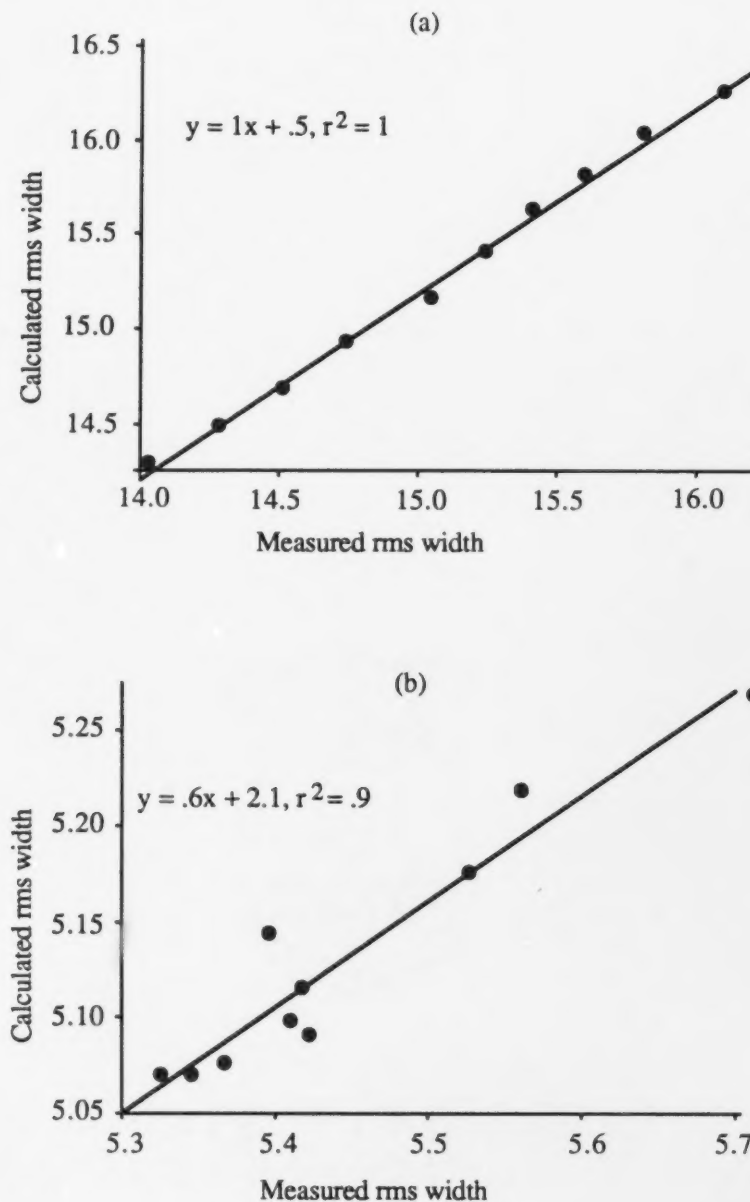


Figure 4.2.5: Linear regression between rms widths of the measured and calculated attenuation profiles for detectors 10 (a) and 22 (b). The units of rms widths are in terms of sample spacing ($4.2.226 \times 10^{-3}$ cm).

A linear regression of the calculated and the measured rms widths was then calculated for each detector and the standard error of estimate (SEE) determined as a percentage of the

mean rms width value from the measured data (Table 4.2.1). Regressions for detectors 10 and 22 are shown in Figure 4.2.5a and 4.2.5b. Detectors 20 and 21 have responses similar to Figure 4.2.5b; all other detectors were similar to the regression shown in Fig 4.2.5a.

There are a number of sources of error in the measurements carried out to validate the model. First, the assumption that the steel wire was placed exactly perpendicular to the slice plane is not strictly true in practice. Second, the width of X-ray beam profile varies from detector to detector, and the same sampling distance for the measured attenuation profiles used for all detector causes errors which vary between detectors. In particular, detectors aligned to the near side of the heel have different rms focal spot widths along the anode and a larger %SEE in the model prediction. Third, the accuracy of measuring the detector angles α_i varies from detector to detector [83]. Despite these errors, the measured attenuation profiles for most of the detectors are in close agreement with the model. Moreover, the coefficient of variation (%cv) of the rms width of the focal spot intensity distribution along the anode given by 21 detectors is 2%. The model can thus be used to describe the X-ray beam profile for the purpose of estimating the PSF of a CT scanner.

4.2.3 Calculation of the PSF

If the beam profile corresponding to each detector in a CT system is known, the data collection system can then be simulated to compute the projections of a point-object. The operation of the special purpose CT scanner [79] was simulated in order to generate projection data for a 0.23 mm diameter circular object positioned at a number of spatial locations aligned with the grid points of the reconstructed image. The scanning procedure for this CT system is a rotate-only type, similar to that shown in Figure 2.1.2. As indicated in section 2.1.1, the fan-beam projections can be re-ordered into parallel beam projections for the purpose of image reconstruction. Each resulting parallel projection, however, contains the number of samples which are equal to the number of detectors in the fan-beam. Since there are only 26 detectors in the fan-beam of this CT scanner, the data collected in a single rotation around the object would provide only 26-point parallel projections after re-ordering. In order to avoid the errors in the reconstructed image due to the under-sampling of the projections, this scanner utilizes a special scanning approach which is different from the usual single rotation scheme in two respects. First, in a single rotation scheme, the scanner is required to rotate only by 180° (+fan angle) to collect equivalent parallel projections over 180° . A further rotation effectively repeats the already known samples if the detectors on both sides of source (focal point) to center-of-rotation (SCR) line are symmetric. In this system, the detectors on either side of the SCR line are made

slightly asymmetric. This results in new sample locations during a rotation of the fan-beam further than 180° (+fan angle). Second, this asymmetry can be varied by linearly shifting the detector assembly with respect to the SCR line in a procedure utilizing multiple rotations. This CT scanner employs 5 rotations of a linearly-shifted, asymmetric fan-beam to generate 130-point fan-beam projections spaced at $2/3^\circ$ over 360° . The fan-beam projections were re-binned as parallel projections; the data between 180° and 360° was folded back between 0° and 180° to provide 270 irregularly spaced parallel projections each of 260-points. Before reconstruction, the irregularly spaced projection data was re-sampled at 257 regular intervals using linear interpolation. Therefore, the re-binned data contains 270 projections each sampled at regularly spaced 257 points. The images of the point-objects located at different pixel positions were then reconstructed using the modified interpolation in the filtered backprojection method, in order to estimate the PSF at each such point. A 15-by-15 pixel area centered at the object position in a 257-by-257 pixel reconstructed image represents the PSF.

In this simulation, the SCR distance, the focal spot to the detector arc distance, the anode heel angle, and the angular orientations of the detectors with respect to the SCR line, were made exactly the same as for the actual CT scanner. The model was used to compute X-ray beam profiles for each of the detectors at specified distances from the focal spot. The exponential edge gradient effect resulting from integrating the photon intensities instead of line integrals [81] was deliberately introduced into the projection data, but non-linearities in photon counting and beam hardening effects [82] were ignored because of the low filament current (2.8 mA, 50 kV) and heavy filtration (0.70 mm copper) used for the X-ray tube. Blurring due to the rotational movement of the scanner was also simulated in that the ray-sum for each detector at a specified angular position of the focal spot was calculated as an average of those within $\pm 1/3^\circ$ of the angular position. The asymmetry and linear shifts of the fan-beam for multiple rotations were closely followed in the simulation. Therefore, if the model for the X-ray beam profiles is valid, then the PSF computed using this data must match the experimental PSF for this scanner at all spatial locations. In addition, the computed PSF is unaffected by misregistration of the point-object on the sampled grid points and the photon statistical noise. Also, by repeating the procedure the PSF can be computed for any number of points located on the grid points.

To directly compare the computed and measured PSFs, projection data for a thin steel wire was collected using the same CT system. The re-ordering and image reconstruction procedures were identical to those used in conjunction with the simulated

data as described in the previous paragraph. A 15-by-15 pixel area of image represented the measured PSF. In Figure 4.2.6, the directly measured PSF at the center of the scan field is compared with the PSF computed using the model. Similarly, in Figure 4.2.7, two PSFs at a distance of 9 pixels (pixel size = 0.34 mm) from the center of the scan field are shown. A visual comparison of the directly measured and computed PSF (Figure 4.2.6) illustrates the errors caused by misregistration of the wire on the sampled grid points, and by measurement noise. In order to emphasize the error due to misregistration, noise was significantly reduced in the measured PSF (Figure 4.2.7) by using a very slow scan speed (1/8 normal). It should be remembered that the PSF can be asymmetric as shown in Figure 4.2.7 (b), however, asymmetry in the measured PSF can be due either to misregistration or to the design of scanner. Because of this asymmetry, image restoration carried out using measured PSFs would thus be inaccurate. Therefore, estimation of the PSF should be carried out in such a way that it is free from noise and asymmetry caused by misregistration, otherwise, extraneous phase correction during the restoration may produce undesirable results.

In order to describe spatial variations in PSF, calculations made at a number of locations in the scan field were compared. For this comparison, the parameters (A_1 , A_2 , A_3) of the 2-D Gaussian function (equation (4.2.21)) were computed using a non-linear least-squares fit of the calculated PSF data to the function

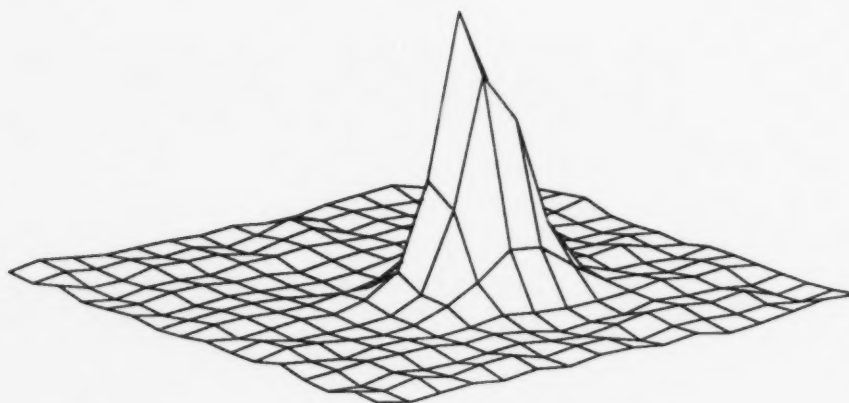
$$F(t,s) = A_1 \exp \left[- \left(\frac{t^2}{2A_2^2} \right) - \left(\frac{s^2}{2A_3^2} \right) \right] \quad (4.2.21)$$

where t , s are the rotated (x,y) coordinates

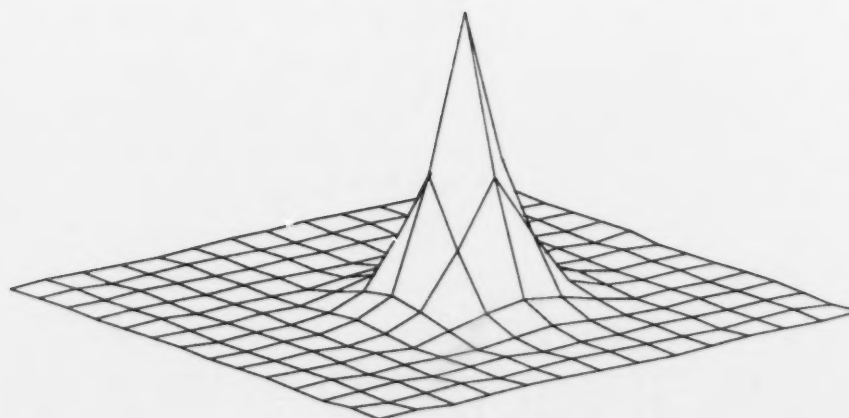
$$t = x \cos \gamma_0 + y \sin \gamma_0 - R_0 \quad (4.2.22)$$

$$s = -x \sin \gamma_0 + y \cos \gamma_0 \quad (4.2.23)$$

(R_0 , γ_0) is the polar form of the Cartesian coordinate (x_0, y_0) of the point-object. It should be noted that the function $F(t,s)$ is required for the (t,s) values corresponding only to the discrete samples of 15-by-15 pixel PSF. The function $F(t,s)$ and the coordinate rotations of equations (4.2.21), (4.2.22) and (4.2.23) were used, as suggested by Verly and Bracewell [71], to demonstrate the rotate-only nature of the PSF due to point-objects located at a fixed R_0 . The results of this parametrization for the PSF due to points located at $R_0 = 0.572$ cm and $R_0 = 3.33$ cm are shown in Table 4.2.2. The parameters A_2 and A_3 change as a function of the radial distance from the origin for fixed γ_0 (Figure 4.2.8).

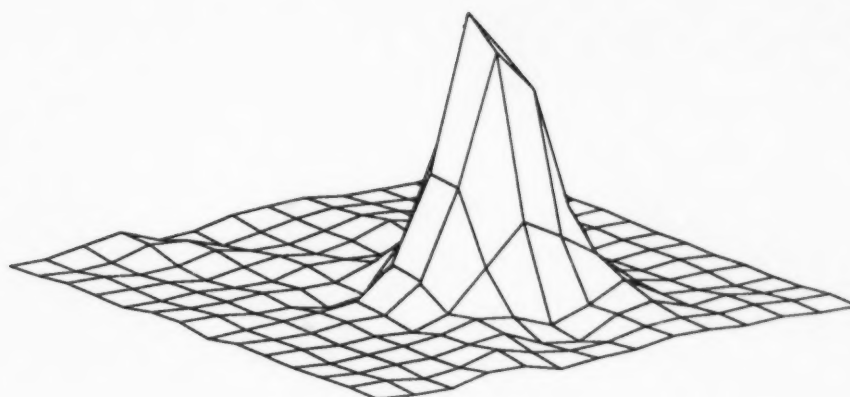


(a)

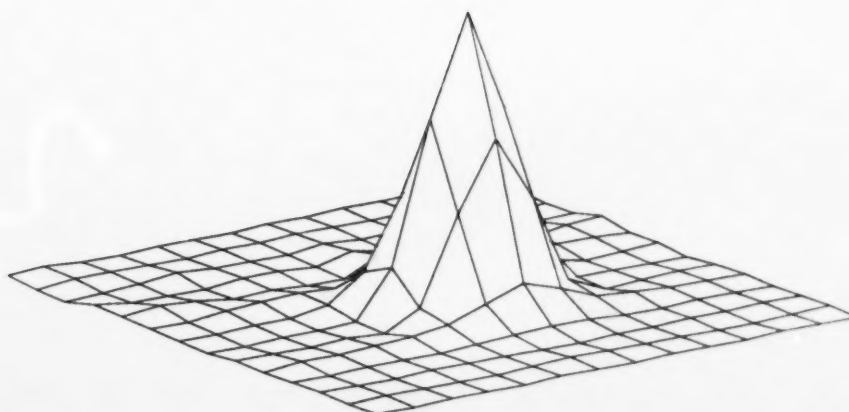


(b)

Figure 4.2.6: Point spread function (a) directly measured by scanning at normal scan speed, and (b) calculated by simulating operation of the CT scanner for a 0.23 mm diameter circular object located at the center of the scan field.



(a)



(b)

Figure 4.2.7: Point spread function (a) directly measured by scanning at 1/8th normal scan speed, and (b) calculated by simulating operation of the CT scanner for a 0.23 mm diameter circular object located at a distance of 9 pixels (pixel size = 0.34 mm) from the center of the scan field.

For the computed PSFs, variations in the spread parameters of the 2-D Gaussian function (Figure 4.2.8) are small due to the small field of view of the scanner. Variations at a constant distance from the origin are negligible, since the PSF should only rotate according to the angular position of the point-object. This can be seen in Table 4.2.2 for both values of the radial distance. The small differences are due to the fact that the reconstruction of images is done on sampled (x,y) coordinates instead of on (t,s) coordinates. It should be noted that these patterns in the PSF may be masked by experimental error if the PSF is measured directly from the wire scans.

γ_0 (Degree)	$R_0 = 0.572$ cm		$R_0 = 3.33$ cm	
	A_2 (pixels)	A_3 (pixels)	A_2 (pixels)	A_3 (pixels)
0	0.963	1.061	0.837	1.092
45	0.985	1.062	0.865	1.104
90	0.963	1.061	0.837	1.092
135	0.985	1.062	0.865	1.104
180	0.963	1.061	0.837	1.092
225	0.985	1.062	0.865	1.104
270	0.963	1.061	0.837	1.092
315	0.985	1.062	0.865	1.104

Table 4.2.2: Parameters A_2 and A_3 , for the parametrized PSF, as a function of angular position of a point-object located at a constant radius from the center of scan field. The units of spread parameters are in terms of pixel size (0.34 mm). The power accounted for the 2-D Gaussian model is 99.7% and 98.7% respectively at distance 0.572 and 3.33 cm from origin.

In Figure 4.2.8, the variation in the parameters of the fitted 2-D Gaussian function to the PSF are shown. The prominent feature in this relationship are the opposite trends in the variations of A_2 and A_3 with respect to radial distance. This finding is similar to that observed by Doré et al [77]. The non-monotonic behavior of the response is due to the discrete nature of the data collection and image reconstruction methods.

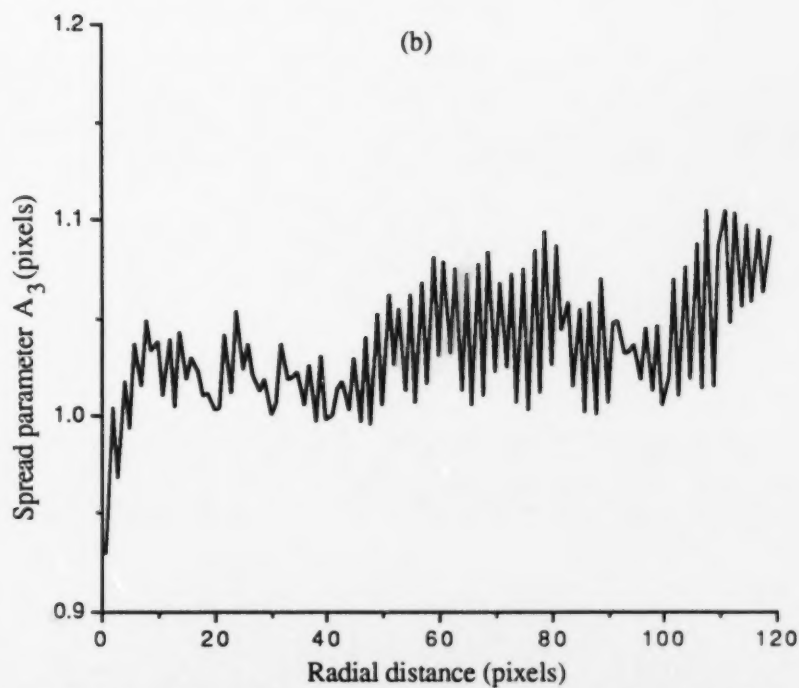
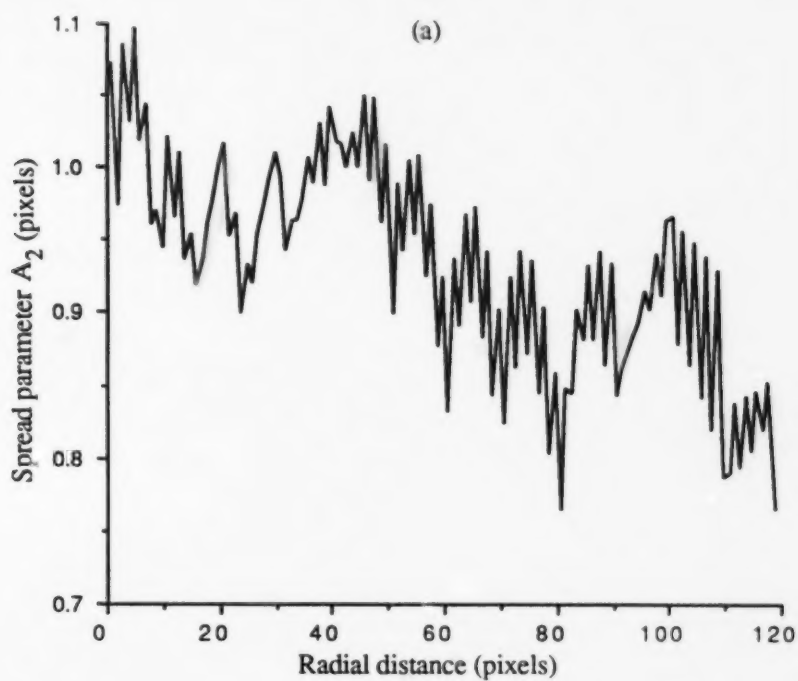


Figure 4.2.8: Variation of the spread parameter (a) A_2 and (b) A_3 for the parametrized Gaussian PSF as a function of the radial distance from the center of scan field. The units of spread parameter and radial distance are in terms of pixel size (0.34 mm).

4.3 SUMMARY

In summary, the PSF of a CT system is spatially variant and is a function of the radial position of the point-object. For points on a circle of a given radius, PSFs are only rotated with respect to the angular position. For image restoration the PSF can be computed using the method described. An accurate knowledge of the X-ray beam profiles allows computation of the PSF without direct phantom measurements. The only measurements required are those of the focal spot intensity and detection sensitivity distributions. If the CT system manufacturer provides this information, along with the usual geometric information, computation of the spatially varying PSF can be carried out and experimental errors resulting from the positioning of phantoms and noise can be avoided.

5. RESTORATION OF EXPERIMENTAL CT IMAGES

In the previous chapter, the spatially variant nature of the PSF of a CT system was discussed. A spatially variant PSF can either be measured by scanning a phantom or estimated from the profiles of the X-ray beam.

There are numerous systems in optical imagery where the PSF is spatially variant. In arial photography of the Earth, Moon, and planets, the spatially variant degradations which limit the achievable spatial resolution in the recorded image, are due to motion blur and geometrical distortion [65, 86]. The PSF becomes spatially variant if the object is not moving in a plane parallel to the image plane, e.g. in the case of circulatory motion where the PSF width increases linearly with the object distance from the optical axis. The most important type of geometrical distortion is due to ray aberration. Due to aberrations, a point-object is relatively displaced in the image plane from its ideal position (i.e. if no distortions were present), and magnified depending upon the geometrical distances of the object and image planes from the lens. The displacements in two directions of the image plane depend upon the location of the point in the object plane. When these displacements are linearly proportional to the radial distance of the point from the optical axis, the geometrical distortion is termed a comma system. In the astigmatism and curvature field resulting from distortion, the displacements are proportional to the second power of the radial distance [65].

In SPECT and PET, reconstructed images have non-stationary resolution for reasons similar to CT. Therefore, development of an image restoration procedure for a spatially variant PSF is important for the other medical imaging modalities as well.

In this chapter, numerous different approaches to the restoration of experimental CT images are described. Image restoration procedures are applied to the reconstructed images in the Cartesian coordinate system as well as to images transformed into polar coordinates. Approximations are suggested to make the PSF spatially invariant. The restored image obtained with this approximation is compared to that obtained with the spatially variant form of the locally adaptive method.

5.1 RESTORATION IN TRANSFORMED COORDINATES

The spatial variation of the PSF in optical images resulting from various motion and geometrical distortions is similar to that in CT images. For example, the PSF resulting from

circulatory motion, comma systems, and astigmatism and curvature of field only rotates according to the angular position of the point-objects located on a circle in the object plane. As a result, the coordinate transformation methods which have been applied to these images, are also applicable to CT images [71].

The basic idea behind the coordinate transformation method is that the reconstructed images in Cartesian coordinates is transformed into another system where the PSF becomes relatively independent of the position [65, 85, 86, 87]. In Cartesian coordinates, the discrete image formation model for a spatially variant blur can be written as:

$$g(i,j) = \sum_{m=-M}^M \sum_{n=-M}^M f(i-m,j-n)h(m,n; i,j) + e(i,j) \quad (5.1.1)$$

where f and g are the original and reconstructed images respectively, and e is the noise in the reconstructed image. The sampled PSF due to a point-object located at (i,j) is denoted by a $(2M+1)$ -by- $(2M+1)$ pixel function $h(m,n; i,j)$.

In order to perform the multiplications inside the double summation of equation (5.1.1), the PSF at each (i,j) must either be stored or calculated in real time. The computation of PSF in real time would be very expensive and the pre-computed PSF would require a large storage space; approximately 2.97 megawords for a 257-by-257 pixel image and a 9-by-9 pixel PSF. In addition, the solution of equation (5.1.1) for $f(m,n)$ would be complicated since it is not a 2-D convolution.

The purpose of coordinate transformation is to reduce both the storage requirements for the pre-computed PSF and the computational requirements for solving the equations describing the image formation model. A polar coordinate transformation method has been used by Sawchuk [65] to restore the astigmatism and curvature of field effects with the solution of the transformed set of equations obtained by using singular value decomposition. In CT images, however, the number of equations is much higher and singular value decomposition is not practical.

5.1.1 Regularly Sampled Polar Coordinates

In section (4.2.3), a 2-D Gaussian function fitted to the computed PSF adequately accounted for the shape of the PSF. The PSFs for the point-objects located on a circle in the scan field were rotated according to the angular position, as shown in the following equations and Figure 5.1.1.

$$h(m,n; i,j) = A_1(i,j) \exp\left(-\frac{t^2}{2A_2^2(i,j)} - \frac{s^2}{2A_3^2(i,j)}\right) \quad (5.1.2)$$

where

$$t = m \cos\theta + n \sin\theta - r \quad (5.1.3)$$

$$s = -m \sin\theta + n \cos\theta \quad (5.1.4)$$

$$\theta = \arctan(j/i) \quad (5.1.5)$$

$$r = \sqrt{i^2 + j^2} \quad (5.1.6)$$

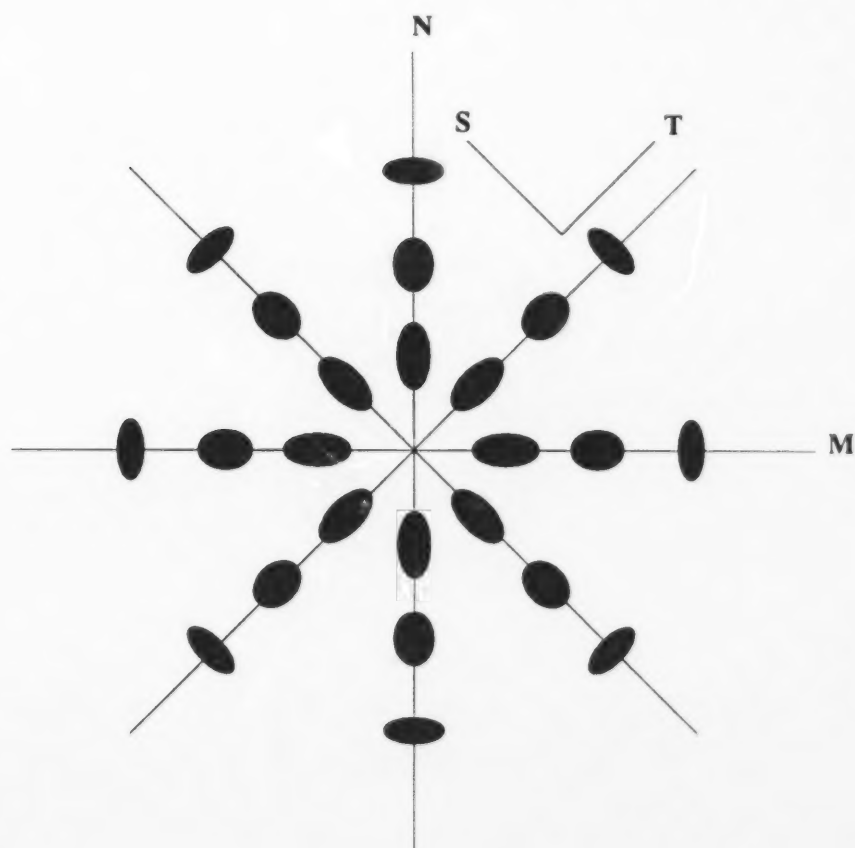


Figure 5.1.1: The PSF at various locations in the Cartesian coordinate system. The widths of the PSF for the point-objects on a circle are the same, and the PSF is rotated by angular position.

It has also been shown in section 4.2.3 that the parameters A_1 , A_2 and A_3 are functions of radial distance r , but are independent of angular position θ . In the following discussion, therefore, the parameters A_1 , A_2 and A_3 are written as a function of r .

Let (r_1, θ_1) be the location in polar coordinates corresponding to the location (m, n) in Cartesian coordinate system such that

$$m = r_1 \cos \theta_1 \quad (5.1.7)$$

$$n = r_1 \sin \theta_1 \quad (5.1.8)$$

A direct substitution of equations (5.1.7) and (5.1.8) into (5.1.3) and (5.1.4) results in the following equations.

$$t = r_1 \cos(\theta - \theta_1) - r \quad (5.1.9)$$

$$s = r_1 \sin(\theta - \theta_1) \quad (5.1.10)$$

Therefore, the PSF, $h_r(r_1, \theta_1; r, \theta)$, in polar coordinates can be written as:

$$h_r(r_1, \theta_1; r, \theta) = A_1(r) \exp\left[-\frac{(r_1 \cos(\theta - \theta_1) - r)^2}{2A_2^2(r)} - \frac{(r_1 \sin(\theta - \theta_1))^2}{2A_3^2(r)}\right] \quad (5.1.11)$$

The PSF in polar coordinates becomes independent of the angular variable since only $(\theta - \theta_1)$ appears in equation (5.1.11). However, the widths of the PSF, both in the radial and angular directions, vary as a function of the radial variable r (Figure 5.1.2).

The reconstructed image $g_r(r, \theta)$ in polar coordinates is related to the original image $f_r(r_1, \theta_1)$ by the following equation:

$$g_r(r, \theta) = \sum_{r_1=-M}^M \sum_{\theta_1=-\theta(r)}^{\theta(r)} f_r(r_1, \theta_1) h_r(r_1; r, \theta - \theta_1) + e(r, \theta) \quad (5.1.12)$$

where $2\theta(r)$ is the angle subtended by the PSF at the origin in Cartesian coordinates. The variables r , r_1 , θ , and θ_1 are discrete, and the transformation from a sampled image in Cartesian coordinates requires some form of interpolation.

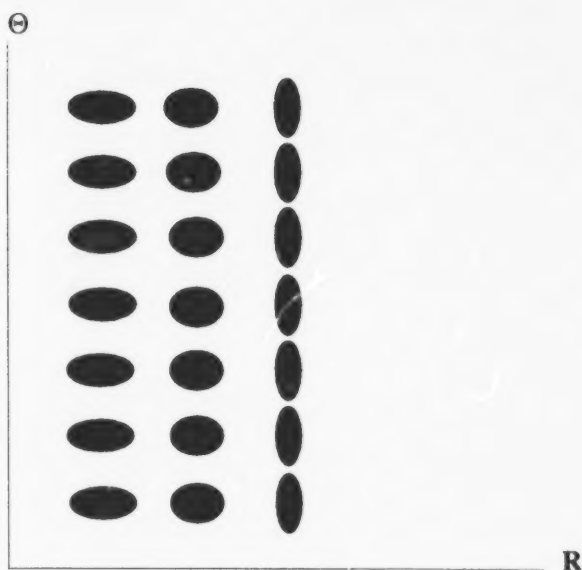


Figure 5.1.2: The PSF in polar coordinate system becomes spatially-invariant for a fixed radial position. At different radial positions, the widths in r and θ directions vary as shown.

Noise in the reconstructed images is spatially correlated (non-white), and if the noise in the projection data is assumed to be spatially stationary then the autocorrelation function of the noise in the reconstructed image is spatially invariant. In the regularly sampled polar coordinate system, the autocorrelation function of noise is spatially variant since regular increments of radial and angular variables result in irregular increments of (i,j) . The autocorrelation function must be estimated at each radial position in polar coordinates to take the noise into account. In addition, the noise in the transformed image cannot be pre-whitened using a spatially invariant filter function. For the sake of simplicity, therefore, the noise in image g_r is assumed to be spatially uncorrelated.

The image formulation model (5.1.12) can be written in matrix form as:

$$\mathbf{G}_r = \mathbf{H}_r \mathbf{F}_r + \mathbf{E}_r \quad (5.1.13)$$

where $\mathbf{F}_r$, $\mathbf{G}_r$ and $\mathbf{E}_r$ are vectors containing the row-wise (column-wise) arranged sequences $f_r(r,\theta)$, $g_r(r,\theta)$ and $e(r,\theta)$ respectively. The matrix $\mathbf{H}_r$ results from similarly arranging the PSF $h_r(r_1; r, \theta - \theta_1)$.

The regularized, weighted least-squares solution of equation (5.1.13) was obtained by minimizing the following function.

$$J(F_r) = \gamma(F_r)^T S_r F_r + (G_r - H_r F_r)^T (D_r)^{-1} (G_r - H_r F_r) \quad (5.1.14)$$

where D_r is the diagonal matrix containing the noise variance at each pixel in the transformed image, and S_r is the diagonal matrix similar to S in section (3.1.2.3).

The normal system of equations resulting from minimizing $J(F_r)$ can be given as the following.

$$(\gamma S_r + (H_r)^T (D_r)^{-1} H_r) F_r = (H_r)^T (D_r)^{-1} G_r \quad (5.1.15)$$

This system of equations was solved using the conjugate gradient algorithm discussed in Appendix III. The parameter γ in equation (5.1.15) was chosen to make $\|H_r F_r - G_r\|^2$ greater than or equal to sum of the noise variances at each (r, θ) . In practice, this choice results in approximately the same value of γ for a particular class of objects.

In this thesis, the transformation of the image from Cartesian to polar coordinates was performed using the bilinear interpolation. In polar coordinates, the radial sample spacing was made equal to the pixel size used in Cartesian coordinates. The number of samples per unit area varies as a function of radial distance. In order to obtain the adequate number of samples far from the origin, the angular sample spacing in polar coordinates, $\Delta\theta$, was computed from:

$$\Delta\theta = \frac{a}{r_{\max}} \quad (5.1.16)$$

where a is the pixel size in the Cartesian coordinate system and $r_{\max}$ is the radius of the scan field. The angular sample spacing in equation (5.1.13) ensures that the arc length between two consecutive angular samples is always less than or equal to the pixel size. The reconstructed image was transformed into polar coordinates using 270 angular samples over 180° . For the 257-by-257 pixel image in Cartesian coordinates, a 257-by-270 ($r=-128$ to 128) pixel image in polar coordinates was then formed. As a result of the polar coordinate transformation, the pre-computed PSFs were stored only for various radial positions. The PSF in polar coordinate system was computed at $r=0$ to 128, with 9 radial samples at each position of r ($M=4$). The number of the non-zero angular samples in the PSFs is a decreasing function of the radial position since $\theta(r)$ is a decreasing function of r . The total storage required for the PSFs, at the above values of the parameters, was equal 29.3 Kwords.

The noise variance in the transformed image was computed by propagating the

noise variance of the Cartesian coordinate image through the polar coordinate transformation and bilinear interpolation. The elements of the diagonal matrix S_r were calculated using the local variance of $g_r(r, \theta)$ and the noise variance as in equation (3.1.30).

5.1.2 Irregularly Sampled Polar Coordinate System

The widths of the PSF in s and t directions change as a function of the radial position of the point object (Figure 5.1.1). As a result, the widths in the r and θ directions change as a function of r in polar coordinates (Figure 5.1.2). An irregular sampling scheme of the polar space was suggested by Doré et al [84] such that the PSF width at different points in the sampled polar space becomes identical. This scheme is based on the following transformations.

Let $FWHM_t(r)$ and $FWHM_s(r)$ represent the full width at half maximum (FWHM) for the PSFs in the t and s directions, respectively, as a function of r . The FWHMs in the r and θ directions, $FWHM_r(r)$ and $FWHM_\theta(r)$, can be computed as:

$$FWHM_r(r) = FWHM_t(r) \quad (5.1.17)$$

$$FWHM_\theta(r) = 2 \tan^{-1} \frac{FWHM_s(r)}{2r} \quad (5.1.18)$$

The sampling intervals in the r and θ directions were made irregular in such a way that the PSF for the resulting sampled image became spatially invariant. Considering that u and v denote the non-uniformly sampled r and θ directions, the relationship between the uv -coordinates and the FWHMs in radial and angular directions are expressed as

$$u(n) = n \Delta u \quad (5.1.19)$$

$$\Delta r[r(u(n))] = \frac{FWHM_r[r(u(n))]}{N} \quad (5.1.20)$$

$$r(u(n+1)) = r(u(n)) + \Delta r[r(u(n))] \quad (5.1.21)$$

$$\Delta \theta[r(u(n))] = \frac{FWHM_\theta[r(u(n))]}{M} \quad (5.1.22)$$

$$v(m) = m \Delta \theta[r(u(n))] \quad (5.1.23)$$

where N and M are the constant number of samples across the PSF width, $FWHM_r(r)$ and $FWHM_\theta(r)$, achieved by varying the sampling intervals Δr and $\Delta \theta$ according to the above

equations.

The main problem with this approach is that a PSF at the position (r_0, θ_0) is sampled according to the sampling intervals Δr and $\Delta \theta$ which vary within the PSF width. Assume that this PSF is non-zero between the radial distances r_1 and r_2 . The sample indices in v -direction at these radial positions can be computed as

$$m_1 = \frac{\theta_0}{\Delta \theta(r_1)} \quad (5.1.24)$$

$$m_2 = \frac{\theta_0}{\Delta \theta(r_2)} \quad (5.1.25)$$

Therefore the line joining the points at r_1 and r_2 at an angle θ_0 is slanted by an amount $(m_1 - m_2)$. Moreover, this slant is directly proportional to the angular position of the PSF. As a result, the PSFs due to the point-objects located on a circle, which were identical in shape in the regularly sampled polar coordinate system, will no longer be identical in shape in uv -plane since the slant $(m_1 - m_2)$ is a function of the angular position. The purpose of the coordinate transformation approach is thus lost.

If the sampling intervals within the width of the PSFs are assumed to be constant then the above approach results in a spatially invariant PSF in uv coordinates. For example, a number of PSFs in Cartesian coordinates are shown in Figure 5.1.5. In Figure 5.1.6, these PSFs are shown in uv coordinates when the sampling intervals are allowed to vary continuously as a function of the radial position. The above mentioned slant in the PSFs and the variations with respect to the angular position are quite evident. If the sampling intervals within the widths of the PSFs are constant then all the PSFs become identical in uv coordinates (Figure 5.1.7). The breaks in the PSFs in Figure 5.1.7 are due to the conflicting angular sampling intervals at the adjacent radial positions.

The idea of the non linear sampling interval originated from the application of coordinate transformation to the comma like degradations [85]. In that case, FWHM_s varied linearly as a function of r which resulted in an angular sampling interval being independent of radial position (equation (5.1.18)). Due to the constant angular sampling interval, the problem of slant was eliminated and the PSF became spatially invariant in uv -coordinates. The variations in the CT PSF do not suggest that the FWHM_s varies linearly with radial position. An irregular sampling geometry in polar coordinates which will make the CT PSF spatially invariant does not exist.

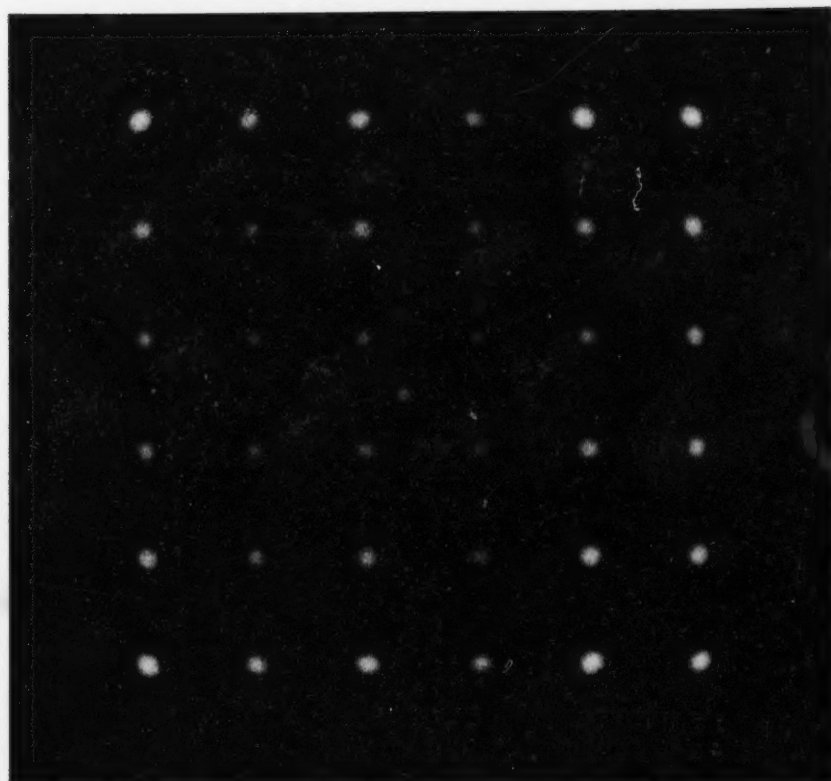


Figure 5.1.5: The PSF at a number of points in Cartesian coordinates.

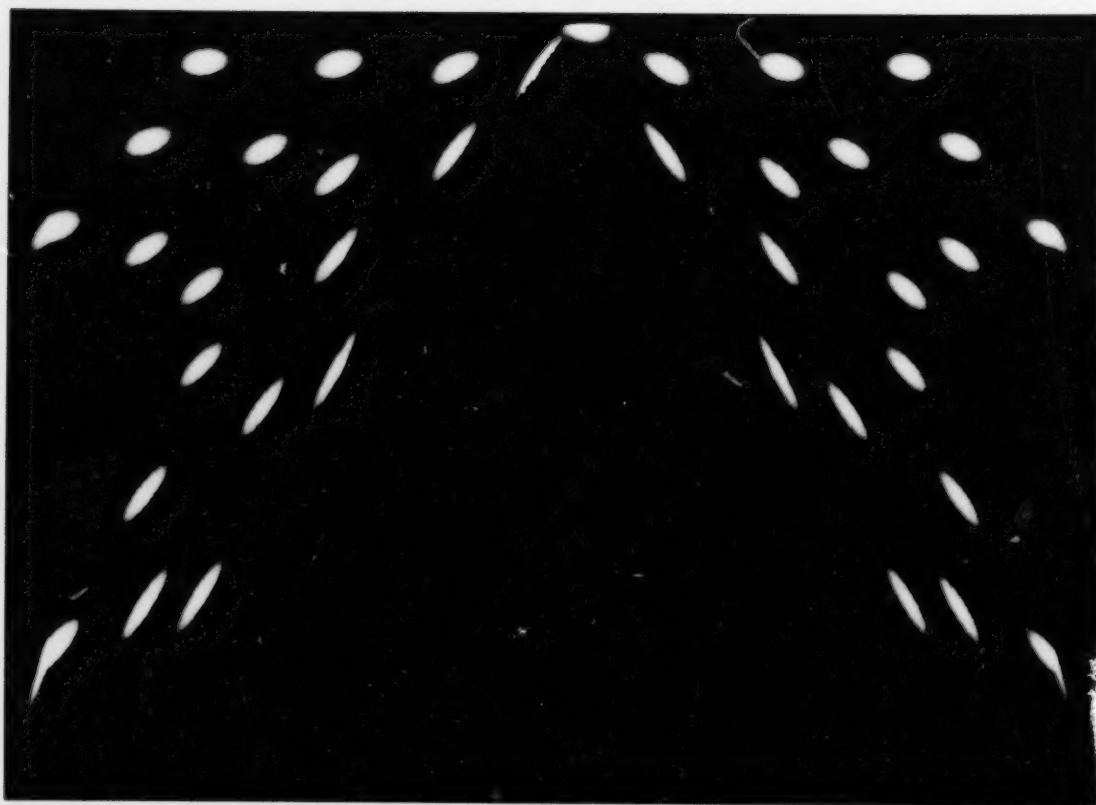
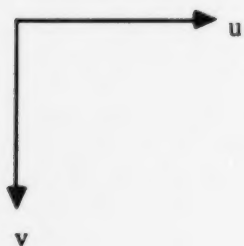


Figure 5.1.6: The PSF corresponding to Figure 5.1.5 in uv -coordinates when the angular sampling interval within the width of the PSF is varying continuously.

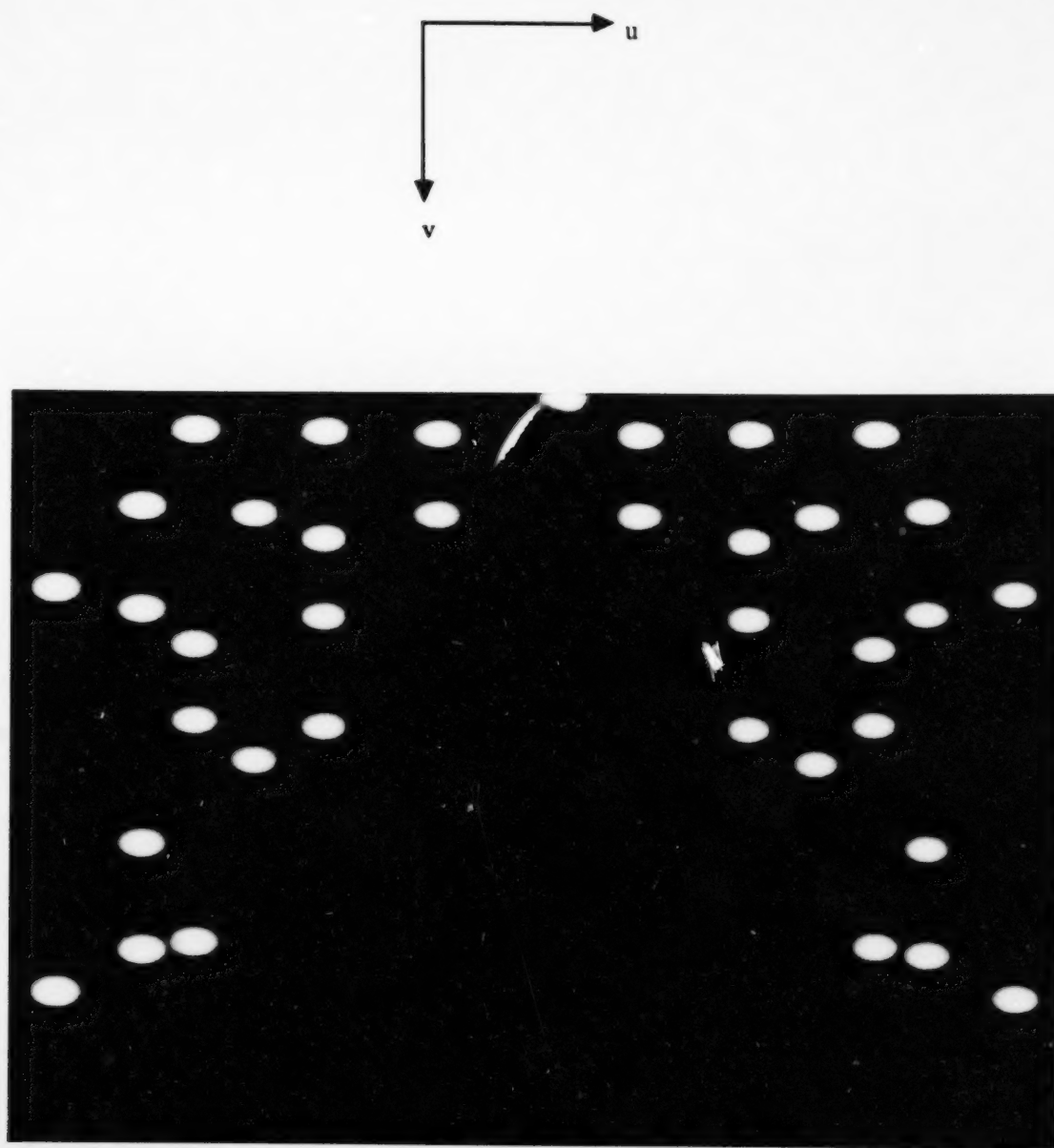


Figure 5.1.7: The PSF corresponding to Figure 5.1.5 in uv -coordinates when the angular sampling interval within the width of the PSF is held uniform.

Therefore, this procedure was not implemented to restore the experimental CT images.

5.2 RESTORATION IN CARTESIAN COORDINATES

In this section, a storage based implementation of the spatially variant locally adaptive method is presented in Cartesian coordinates. Moreover, the restored image after assuming that the PSF is spatially invariant is compared with the restored images resulting from the spatially variant implementation.

5.2.1 Spatially Variant Restoration

The normal equations resulting from the locally adaptive restoration in Cartesian coordinates were given by equation (3.1.31) as:

$$[\gamma S + (\Phi H)^T D^{-1} \Phi H] F = (\Phi H)^T D^{-1} \Phi G \quad (5.2.1)$$

where D is the diagonal matrix of the noise variance for each pixel in the reconstructed image. The matrix D^{-1} is modified from the inverse of matrix D such that the elements corresponding to pixels outside the boundary of object are set to zero. The diagonal matrix S contains elements which are inversely dependent on the variance of the reconstructed image in a 5 by 5 pixel window centered at each pixel. The matrix Φ is the pre-whitening filter derived from the NPS. The operation of this filter on an image vector represents a 2-D convolution since the noise is assumed to be stationary. On the other hand, the operation of the matrix H on an image is not a convolution because of the spatially variant PSF. It should be noted that any coordinate transformation which changes the operation of H to a convolution, will also alter the operation of Φ such that it is no longer a convolution. The operation of H on an image vector, i.e. the computation of the expression HF , can be expressed in a form of the double summation given in equation (5.1.1).

Assume that the PSF at a given point is of size P -by- P pixels ($P=2M+1$) in a reconstructed image of size N -by- N pixels. To compare the storage and computational requirements for different implementations of equation (5.1.1) with those for a spatially invariant PSF, assume that a hypothetical spatially invariant PSF is also of size P -by- P . In equation (5.1.2), the PSF at a particular point (i,j) is the same for (t,s) and $(-t,-s)$; this implies that the PSF $h(m,n ; i,j) = h(-m,-n; i,j)$. Assume that the hypothetical spatially invariant PSF has the same symmetry. For the spatially invariant PSF, the double summation given in equation (5.1.1) is then performed in $N^2 P(M+1)$ multiplications and

$N^2PM + N^2(P(M+1)-1)$ additions.

For the spatially variant PSF, three methods for computing equation (5.1.1) are compared. These are: the direct method; rotation of the PSF using bilinear interpolation; quadrant symmetry method.

5.2.1.1 Direct Method

The PSF $h(m,n; i,j)$ can be pre-computed and stored in order to implement equation (5.1.1) directly. Since $h(m,n; i,j) = h(-m,-n; i,j)$, $N^2P(M+1)$ words of storage are required. The double summation can then be performed in $N^2P(M+1)$ multiplications and $N^2PM + N^2(P(M+1)-1)$ additions. This method has not been implemented since, for $N=257$ and $M=4$, 2.97 megawords of storage are required. The number of multiplications (3×10^6) and additions (5.3×10^6) are the same as the requirements for a spatially invariant PSF implementation.

5.2.1.2 Rotation of the PSF using Bilinear Interpolation

As described previously, the PSF is spatially variant in Cartesian coordinates. If (i,j) are changed to (r,θ) , then the PSF for a constant r has the same shape for all θ but is rotated through angle θ . In this implementation of equation (5.1.1), the pre-computed $h(m,n; i,j)$ is stored for only $N/2$ values of r at an angle 0° . At any other angle, the PSF can be rotated. The storage requirement is significantly reduced; however, the computational requirement is increased since the rotation of the sampled PSF requires interpolation. For a bilinear interpolation used to rotate the PSF, the values of r , $\cos\theta$, $\sin\theta$, and the t and s coordinates corresponding to any one corner of the PSF at a particular (i,j) are required. The value for r for each (i,j) is required to select the parameters $A_1(r)$, $A_2(r)$ and $A_3(r)$. Only the corner values of t and s are required since they can be recursively computed if $\cos\theta$ and $\sin\theta$ are known. For N^2 values of (i,j) , $5N^2$ words to store the pre-computed values of r , $\cos\theta$, $\sin\theta$ and the initial values of the t and s coordinates are required. At a particular value of (i,j) , the computation of the rotated $h(m,n; i,j)$ using the bilinear interpolation then requires 3 multiplications [88] for each (m,n) . One additional multiplication is needed for the product of $f(m,n)$ and the rotated $h(m,n; i,j)$. The total number of multiplications required to implement equation (5.1.1) is then $4N^2P(M+1)$.

For each (i,j) , a recursive calculation of $P(M+1)$ values of t and s requires only $2P(M+1)$ additions. In order to calculate $P(M+1)$ values of the rotated $h(m,n; i,j)$, a total of $6P(M+1)$ additions are required since the bilinear interpolation requires 6 additions [88] for

each (m,n) . In order to utilize the symmetry $h(m,n; i,j)=h(-m, -n, i,j)$, $P(M+1)$ additions are required to perform $\{f(i-m,j-n)+f(i+m,j+n)\}$. After multiplying f 's with corresponding h 's, a total of $P(M+1)-1$ additions are needed. Equation (5.1.1) can therefore be evaluated by $10N^2P(M+1)$ additions.

For $N=257$ and $M=4$, the storage requirement for this method is 0.330 megawords. The number of multiplications and additions are approximately 11.9×10^6 and 29.7×10^6 respectively. This approach to compute the double summation in equation (5.1.1) is useful when the storage required for the pre-computed PSF is not available. However, the required number of multiplications in this implementation is four times larger than in the direct method. The storage requirements of the pre-computed PSF in the direct method is reduced by utilizing the symmetry of the PSF as described in the following section.

5.2.1.3 Quadrant Symmetry Based Approach

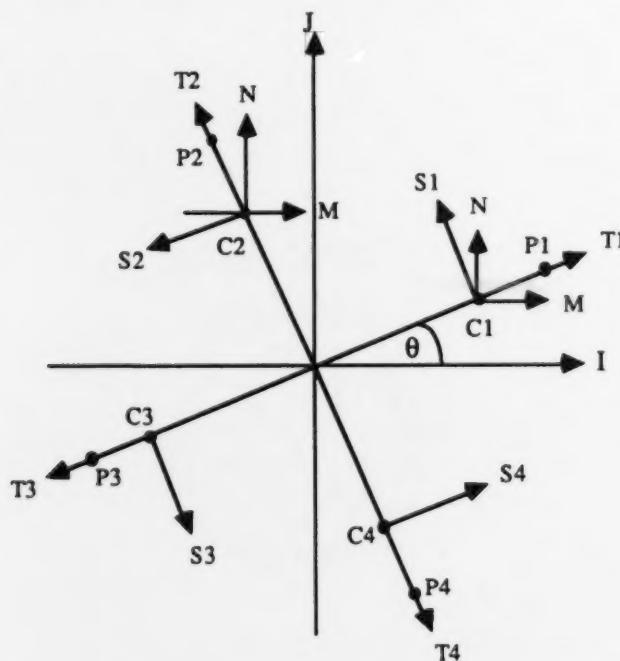


Figure 5.2.1: Quadrant symmetry of the PSF; the PSF at points P1, P2, P3 and P4 is the same since the radial distance, with respect to I-J coordinate system, of points C1, C2, C3 and C4 is identical, and the corresponding angular positions differ only in integer multiple of 90° (see text).

Three coordinate systems are shown in Figure 5.2.1: (1) the fixed I-J coordinates,

(2) the rotating T-S coordinates and (3) the moving origin M-N coordinates. The system M-N is parallel to I-J coordinates, but its origin is relocated to the point at which the PSF is computed. The system T-S is generated by rotating the corresponding M-N system by the angle θ . The points C1, C2, C3 and C4 are located at the same distance r from the origin of the I-J coordinate system. Therefore, the PSF at these points have identical shapes but are rotated with respect to each other. The lines joining the origin to the points C1, C2, C3 and C4 make angles with respect to the positive I-axis denoted as θ , $90^\circ + \theta$, $180^\circ + \theta$ and $270^\circ + \theta$ respectively.

Let us assume that (m,n) are the coordinates of point P1 in the M-N coordinate system for which the (t,s) values are given by the equations (5.1.3) and (5.1.4). The coordinates of point P2 in the coordinate system M-N, relocated at C2, are given by $(-n, m)$. The coordinates of P2 in T2-S2 system are given by

$$t = -n \cos (90+\theta) + m \sin (90+\theta) - r$$

or

$$t = n \sin \theta + m \cos \theta - r \quad (5.2.2)$$

Similarly

$$s = n \sin (90+\theta) + m \cos (90+\theta)$$

or

$$s = n \cos \theta - m \sin \theta \quad (5.2.3)$$

Comparison of equations (5.2.2) and (5.2.3) with equations (5.1.3) and (5.1.4) reveals the same values of (t, s) for points P1 and P2. Therefore, when considered in the M-N coordinate system, the PSF located at the point C2 is the same as that located at point C1 with its m and n coordinates interchanged. Similarly, it can be shown that the PSF at point C3 is the same as that point C1 with the opposite ordering of both m and n indices. Using the same argument, the PSF at C4 is the same as the PSF at C2, therefore, only the PSF at C1 needs to be stored. This suggests that a pre-computed PSF is needed only for one of the quadrants of the image; the PSF in the other three quadrants can then be obtained from the symmetry.

The storage and computational requirements for this implementation are estimated

as follows. For each (i,j) , $P(M+1)$ words are stored resulting in $N^2P(M+1)/4$ words of total storage. The number of multiplications and additions are the same as in the direct method, however, for $N=257$ and $M=4$, the storage requirements are reduced to approximately 743 Kwords.

To this point we have assumed that an image contains N^2 pixels. However, not all of the N^2 pixels in a CT image have to be processed since the field of view of a CT scanner is not a square. Assuming that the scan field is a circle of diameter of N pixels, the total number of pixels within the scan field is $\pi N^2/4$. Therefore, the number of multiplications, additions and storage locations required for the three analytical approaches is reduced by a factor of $\pi/4$. Thus, the storage requirement for the quadrant symmetry method is reduced to less than 584 Kwords.

The quadrant-symmetry based method was implemented to restore images in the Cartesian coordinate system. The solution of equation (5.2.1) was obtained iteratively by using the conjugate gradient algorithm. The multiplication of the matrix $[\gamma S + (\Phi H)^T D^{-1} \Phi H]$ with the restored image vector after each iteration is required for the purpose of computing the incremental improvement in the restored image (Appendix III). The multiplications HF and $H^T F$ can be performed with the identical number of multiplications to that of a spatially invariant implementation since the matrix H is pre-computed and stored. The spatially invariant PSF of the pre-whitening filter was derived from the measured NPS as described in section 5.3. The multiplication of the pre-whitening PSF matrix Φ with the image vector was performed by using the 2-D FFT.

Ideally, the parameter γ in equation (5.2.1) should be chosen such that the residual signal $(G - HF)$ has the same statistics as the noise in CT image. However, the autocorrelation function of the residual signal need to be computed for this comparison which requires a large number of multiplications. In this implementation of the locally adaptive method, an initial restored image was obtained for a value of γ determined as the ratio of noise and signal variances in the reconstructed image G . The noise variance and the signal variance were computed by using the pixels outside and inside the estimated object boundary. The value of γ was then varied such the restored image satisfied the constraint

$$\|G - HF\|^2 = \|D\|^2 \quad (5.2.4)$$

In practice, the noise variances can only be estimated from the measured transmitted photon intensities; therefore, this constraint was only approximately satisfied.

For a particular choice of γ , only 8-10 iterations of the conjugate gradient method were necessary for convergence and this required only 5 to 6 minutes using a SUN SPARC 2 workstation. Moreover, for a particular class of original objects, the optimal value of γ did not change much from image to image so that the double iteration could be avoided.

5.2.2 Spatially Invariant Restoration

In the Cartesian coordinate system, a spatially invariant approximation to the PSF can be obtained if the objects to be scanned occupy only a small region of the scan field. The X-ray beams in a CT scanner diverge or converge from the focal spot to the detector aperture. If the size of the object intercepting the beam is very small compared to the distance between the focal spot and detector aperture, then the X-ray beam can be assumed to be non-divergent within the object. In such cases, the X-ray beam profiles B_0 , defined in the previous chapter, can be assumed to be independent of the distance from the focal spot. Such an assumption would result in a spatially invariant PSF for a single detector scanner, similar to the mathematical simulations presented in chapter 3. Similar arguments were used by Joseph and Spital [81] in order to design a correction procedure for the exponential edge gradient effect.

For a multi-detector fan-beam scanner, assumption of a spatially invariant PSF can only be made after studying the variations in the PSF within the region of interest. In particular, the following observations must hold for spatially invariant image restoration to be reasonable.

(i) The PSF should approximately be circularly symmetric so that the rotation of the PSF in the Cartesian coordinate system is not required. In the view of the parametrization of PSF by using a 2-D Gaussian function, this means that the parameters A_2 and A_3 must approximately be the same in the region of interest.

(ii) The variation in the PSF parameters, A_1 , A_2 , and A_3 , must be very small within the region of interest.

Assuming that A_2 and A_3 are the same and independent of radial position r , the PSF $h(m,n; i,j)$ can be re-written as the following.

$$h(m,n; i,j) = \exp \left[-\frac{(i-m)^2 + (j-n)^2}{2A^2} \right] \quad (5.2.4)$$

Equation (5.2.4) is derived from equations (5.1.2), (5.1.3) and (5.1.4) after assuming that A_1 is normalized to unity, and $A_2(r)=A_3(r)=A$. Therefore, the PSF is spatially invariant within the region of interest since it only depends upon the differences $(i-m)$ and $(j-n)$.

The validity of the above approximations was verified from Figure 4.2.8 in chapter 4. Within a circle of radius of about 60 pixel units, the parameters A_2 and A_3 vary approximately between 0.975 pixel units and 1.05 pixel units. Therefore, a Wiener filter restoration of the images which are limited to this region was deemed reasonable. Examples of this restoration are given in the next section after discussing the NPS measurement.

5.3 NPS MEASUREMENT

A noise pre-whitening filter requires knowledge of the NPS. In simulations of various restoration algorithms in chapter 3, the NPS was calculated using a modified form of the analytical formulation given by Kiejewski and Judy [47]. For an experimental CT scanner, however, the same approach cannot be used. There are two reasons for this: first, it is not known whether the data collected by different detectors in a fan-beam are statistically independent; second, if the reconstruction algorithm is based on parallel projections, re-ordering of the fan-beam into regularly sampled parallel projections requires interpolation, and noise present in the projections becomes spatially correlated in this process. The analytical evaluation of NPS is, however, based upon the assumption of statistical independence of the projection noise. Also, the analytical formulation for NPS computation is based on parallel projections, and it cannot be used if a direct fan-beam reconstruction [37] is employed to avoid the initial interpolation.

The NPS of the X-ray fan-beam CT system was measured as follows. A 6.5 cm diameter water phantom (diameter of the scan field 8.6 cm) was sequentially scanned 140 times at the same site, using a special purpose fan-beam CT scanner [79]. A total of 70 noise images were obtained by subtracting consecutive pairs in the 140 reconstructed images (256-by-256 pixels). A 128-by-128 pixel image, with origin at the center of intensity, was then extracted to ensure that the sub-image was contained within the water region. The sub-image was multiplied by a 2-D Hanning window and a 256-by-256 point 2-D DFT of the resulting image was computed. The squared magnitude of the 2-D DFT was averaged over all 70 images. This result was used as the NPS of the system (Figure 5.3.1)

For three constant radial distances from the origin, 1-D slices through the NPS as a

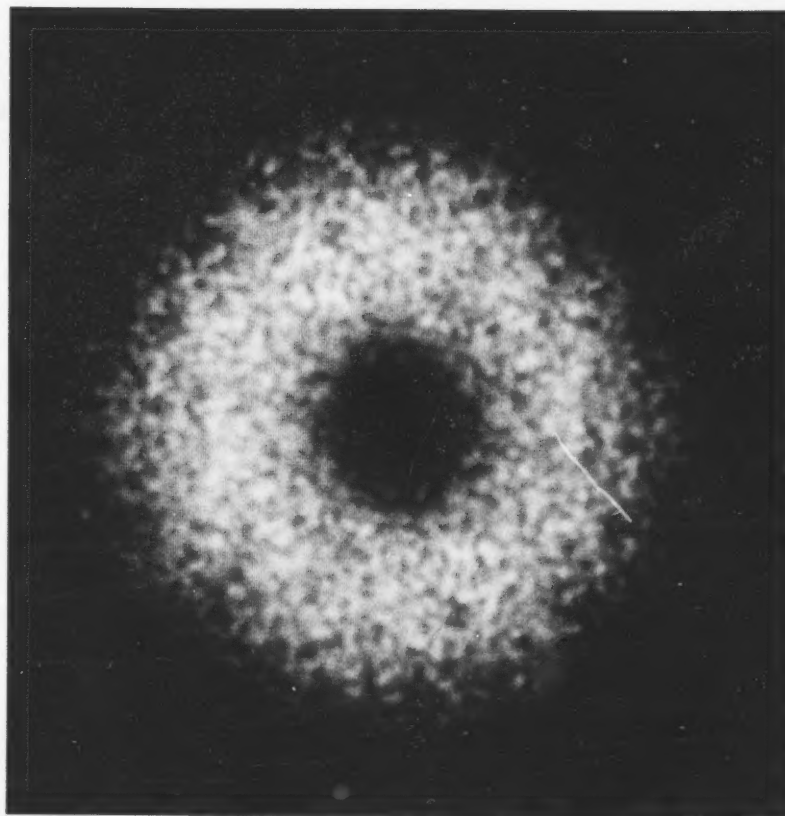


Figure 5.3.1: Measured NPS for the special-purpose CT system.

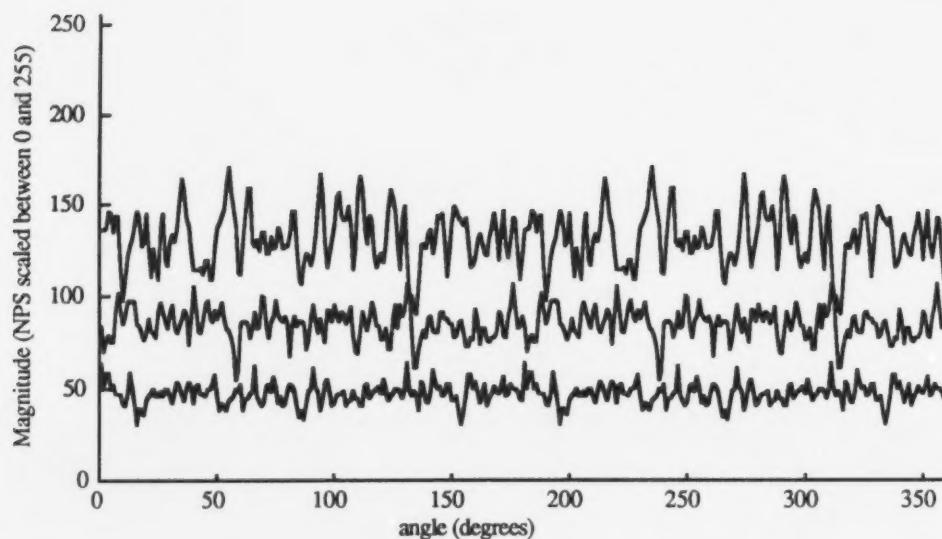


Figure 5.3.2: Magnitude of NPS as a function of angle for constant radial frequencies: (lower) 1.41 cycle/mm, (middle) 1.17 cycle/mm, and (upper) 0.94 cycle/mm. Half the sampling frequency = 1.5 cycle/mm.

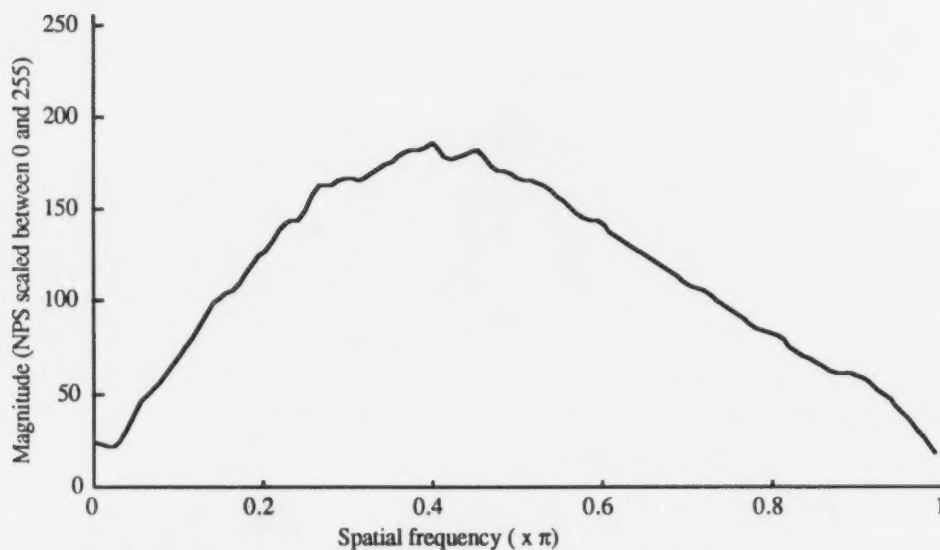


Figure 5.3.3: Magnitude of NPS as a function of radial spatial frequency averaged over 360° from Figure 5.3.1.

function of angle are shown in Figure 5.3.2. In contrast to the measured NPS for a

Technicare scanner [47], the NPS for the special purpose [79] scanner is circularly symmetric.

It should be noted that the principal cause of circular asymmetry in the measured NPS is the 2-D aliasing which occurs due to the non band-limited nature of the interpolation used before backprojection [47]. In the reconstruction algorithm used here, 2-D aliasing was reduced by using a two-step interpolation discussed in chapter 3, section 3.1.2.1.

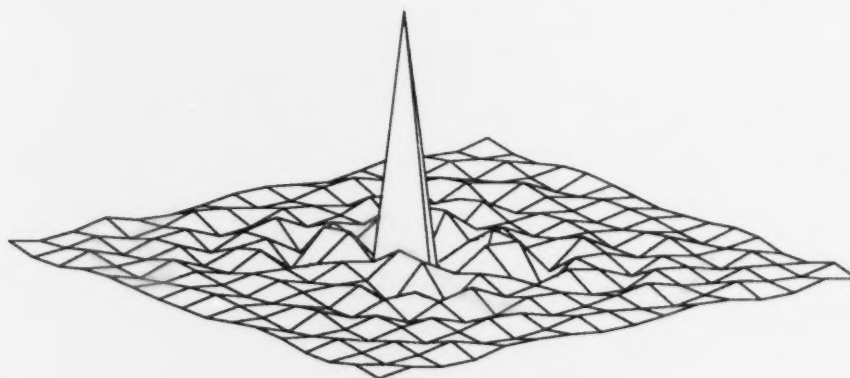


Figure 5.3.4: PSF of the pre-whitening filter calculated from the measured NPS, see text (the value at the origin is scaled to show details).

A 1-D radial NPS calculated by averaging the radial slices through the 2-D NPS for 180 angular positions is shown in Figure 5.3.3. For the angular positions other than 0° and 90° , the radial positions at regular spacings do not always coincide with the pixel locations of the 2-D NPS. At such positions, a bilinear interpolation was carried out to compute the values of the radial slice from the nearest four pixel values of the 2-D NPS. Since the 2-D NPS is circularly symmetric, a 2-D smoothed NPS in sampled Cartesian frequency space was calculated from the 1-D radial NPS. The value of the smoothed 2-D NPS at a particular pixel is calculated from the 1-D radial NPS at a position given by the radial distance of the pixel from the origin. Since the radial distance of the pixel from the origin may not coincide with the points at which the 1-D radial NPS was computed, a linear interpolation between

two neighboring samples of 1-D radial NPS was used. The 2-D Fourier transform of the the pre-whitening filter was then calculated by taking the reciprocal of the square root of the smoothed NPS. It was assumed that noise present at radial spatial frequencies higher than half the sampling frequency (1.5 cycles/mm) was negligible. The 2-D Fourier transform of the pre-whitening filter at these frequencies was assumed to be equal to its value at 1.5 cycles/mm. The PSF of the pre-whitening filter in the spatial domain was then computed as the inverse 2-D DFT of this result. The PSF of the pre-whitening filter is nearly zero at distances far from the origin. A 15-by-15 pixel region of the resulting PSF was used to approximate the pre-whitening filter (Figure 5.3.4).

5.4 RESTORATION RESULTS

A resolution phantom and the distal end of a cadaver femur were scanned by using the special purpose CT scanner [79]. The reconstructed images of the resolution phantom and two slices at the distal end of the cadaver femur are shown in Figures 5.4.1, 5.4.2 and 5.4.3 respectively.

The resolution phantom was chosen to demonstrate the improvement in spatial resolution. In this phantom, the white background corresponds to plexiglass and the black spots to the holes in the plexiglass. The various hole diameters are given in the caption for Figure 5.4.1. The circular ring artifacts in this image are due to the non-uniform detector response within the fan-beam [89]. In particular, these rings appear because the number of photons measured by a detector is not a linear function of the number of photons incident on the detector. This is due to the fluorescence decay time of NaI crystals, and the propagation delay in the photomultiplier and pre-amplifier circuits. The nonlinear response was corrected by fitting a second order polynomial to the detector's response curve.

The reconstructed image of the resolution phantom was transformed into the regularly sampled polar coordinates to perform image restoration. The restored image was then transformed back into Cartesian coordinates as shown in Figure 5.4.4. The smoothness of this image is due to the interpolation required for these coordinate transformations. Justification for the assumption made in section 5.1.1 that the noise in the regularly sampled polar coordinate image is spatially uncorrelated can now be given. In sections (3.1.2.3 and 3.1.2.4), it was shown that the autocorrelation function provided a term in the restoration formulation which smooths the restored image. The images restored in the regularly sampled polar coordinates are smoothed due to the interpolation required to do the coordinate transformation.

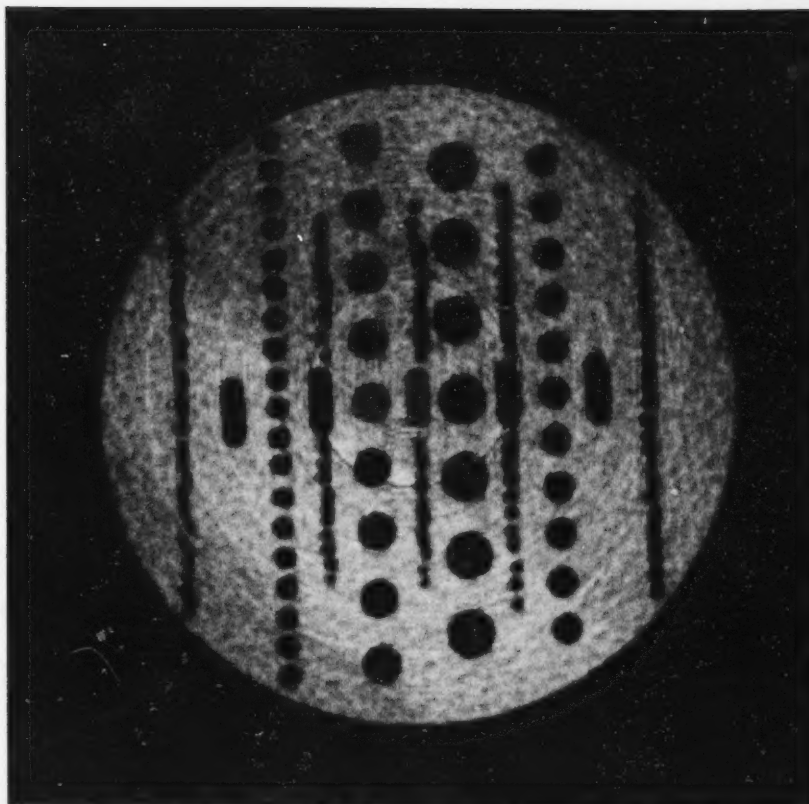


Figure 5.4.1: A CT image of the resolution phantom of plexiglass reconstructed using the filtered backprojection method. The hole diameters from left to right are 0.52, 1.46, 0.68, 3.2, 0.52, 3.8, 0.78, 2.3 and 0.78 mm. In each case, the spaces between holes are equal to the hole diameter. Due to mechanical drilling problems, the small diameter holes (0.52 and 0.68 mm) are not all equidistant.

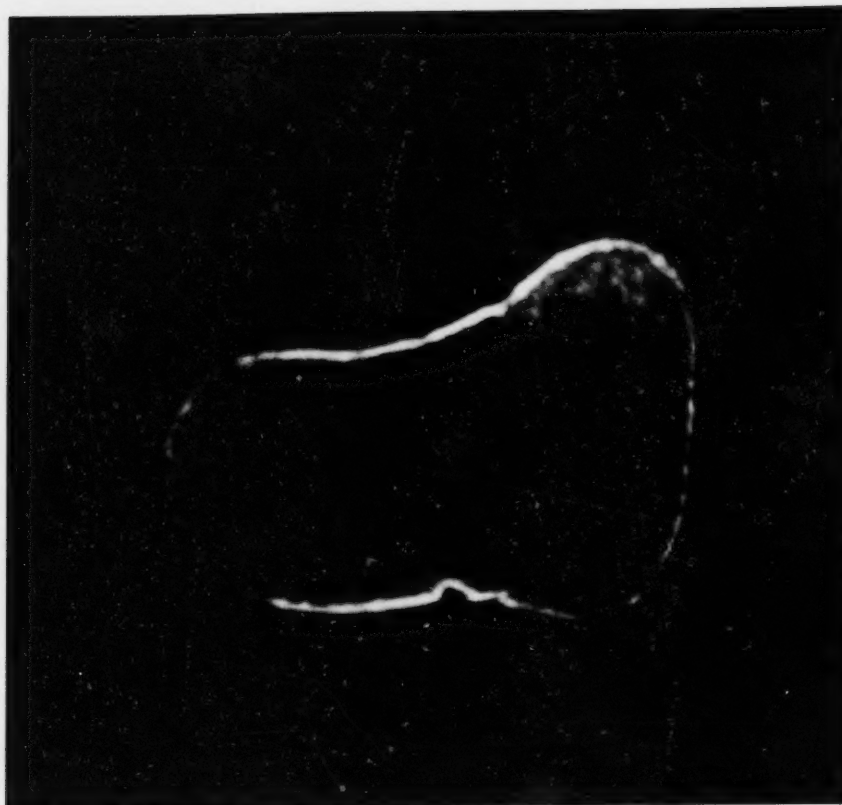


Figure 5.4.2: The reconstructed CT image of the cadaver femur at a distance of 4.5 cm away from the distal end.

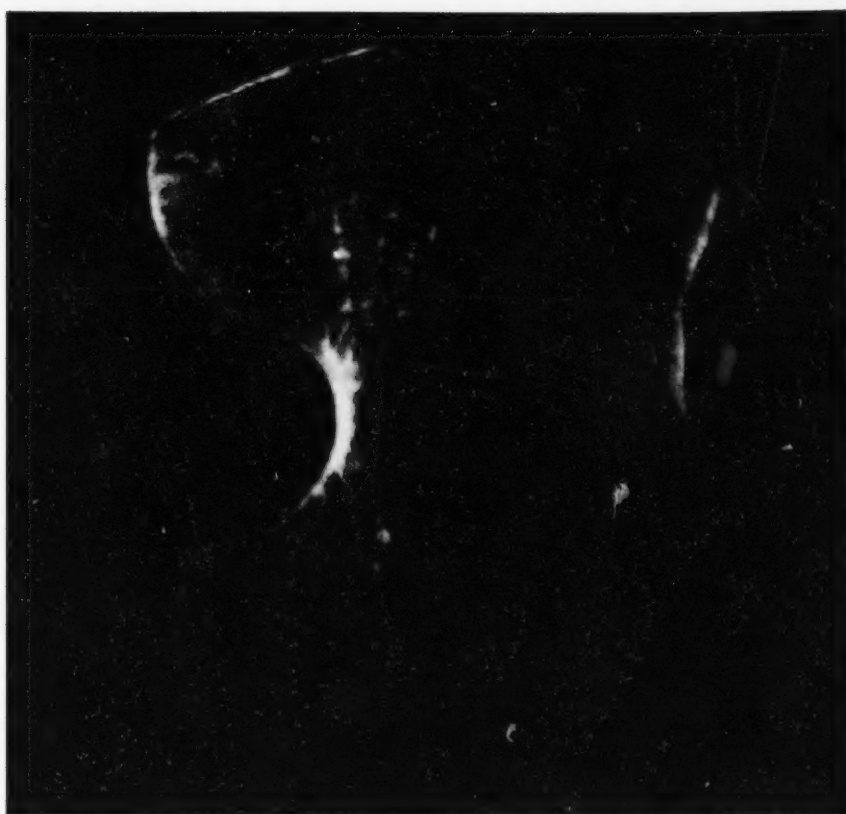


Figure 5.4.3: The reconstructed CT image of the cadaver femur at a distance of 2.5 cm away from the distal end.

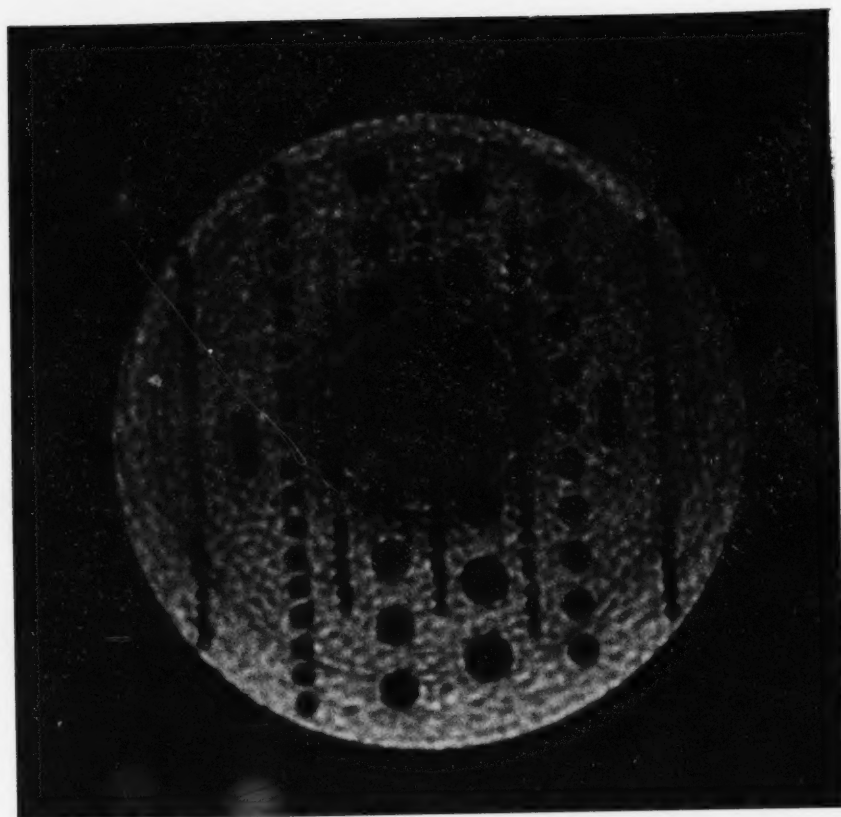


Figure 5.4.4: The restored image of the resolution phantom in the regularly sampled polar coordinates.

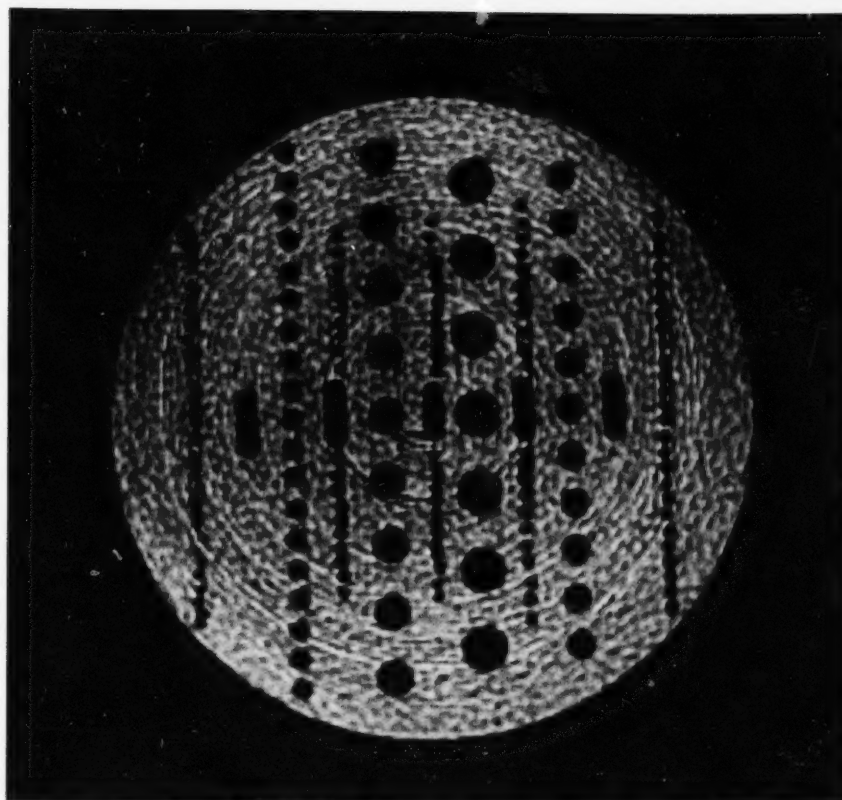


Figure 5.4.5: The restored image of the resolution phantom using the locally adaptive method in Cartesian coordinates

A restored image of the resolution phantom using the spatially variant implementation of the locally adaptive method is shown in Figure 5.4.5. The fine details in this image are more obvious than in Figure 5.4.4 because the restoration is performed in Cartesian coordinate system and the interpolation required to make the coordinate transformation is avoided.

The cadaver femur was chosen to demonstrate that the trabecular bone structure is better visualized by using image restoration methods. The images at two different positions of the femur (Figure 5.4.2 and 5.4.3) are used to show that the spatially invariant approximations to the PSF are good only if the scanned objects occupy a small region of the scan field. The restorations of the bone image in Figure 5.4.2 performed using the quadrant-symmetry based implementation of the locally adaptive method and the Wiener filter are shown in Figures 5.4.6 and 5.4.7, respectively. The PSF for the Wiener filter restoration procedure was computed by taking the average of the parameters A_2 and A_3 over a circle of radius of 60 pixel units. It is clear that the Wiener filter provides equivalent restoration to the locally adaptive method if the image occupies a small region of the scan field. If the scanned objects occupy the complete scan field, then the Wiener filter would be inferior to the locally adaptive method since the approximations for spatially invariant PSF are no longer valid. In order to show this, the distal end of the cadaver femur was again scanned at a position where the size of the bone completely fills the scan field (Figure 5.4.3). The restored images using the Wiener filter and the locally adaptive method are shown in Figures 5.4.8 and 5.4.9. A very noticeable feature in these images is the non-uniformity of the spatial resolution in the restored image obtained by using the Wiener filter since a single PSF used in the procedure inadequately accounts for the significant variations in the width of the PSF over the entire scan field.

In summary, it can be concluded that the locally adaptive restoration procedure based upon the quadrant-symmetry implementation in the Cartesian coordinate system is the best technique in terms of the improvements in the spatial resolution. Due to the interpolation required, the restored images in the regularly sampled polar coordinates are overly smooth. The spatially invariant approximations are not always good. However, if the scanned objects are limited to a smaller region, then the image restoration based on the Wiener filter may provide improvements which are equivalent to the locally adaptive method. In terms of the storage and computational requirements for implementing the restoration procedures, the Wiener filter outperforms every other method since it is based upon a spatially invariant approximation of the PSF. In addition, it is an analytical method

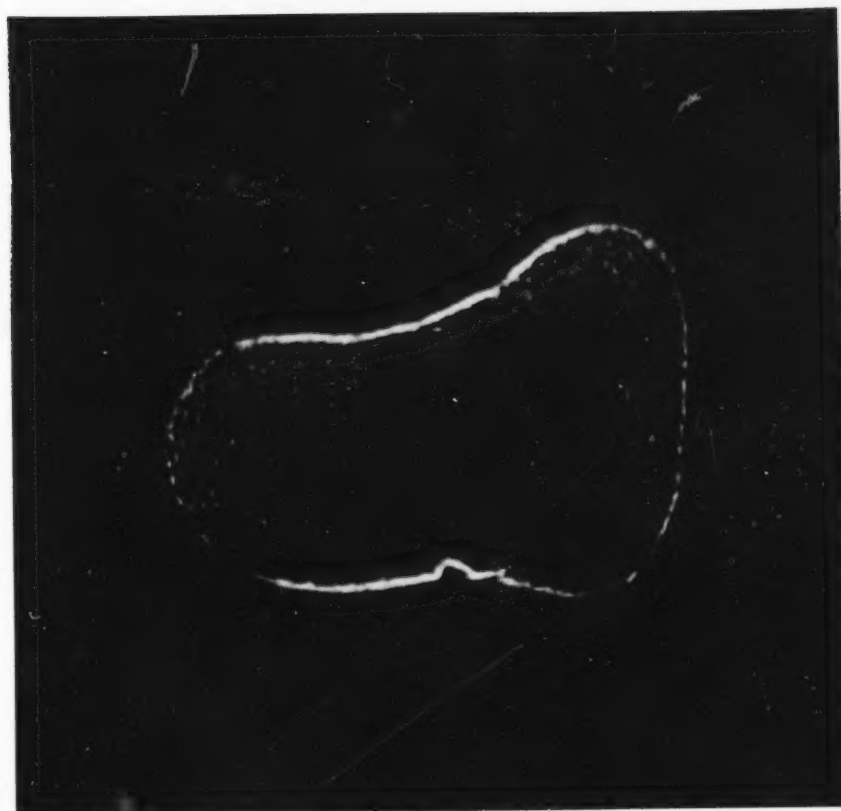


Figure 5.4.6: The restored image of the cadaver femur at a distance of 4.5 cm away from the distal end, using the locally adaptive method in the Cartesian coordinates.

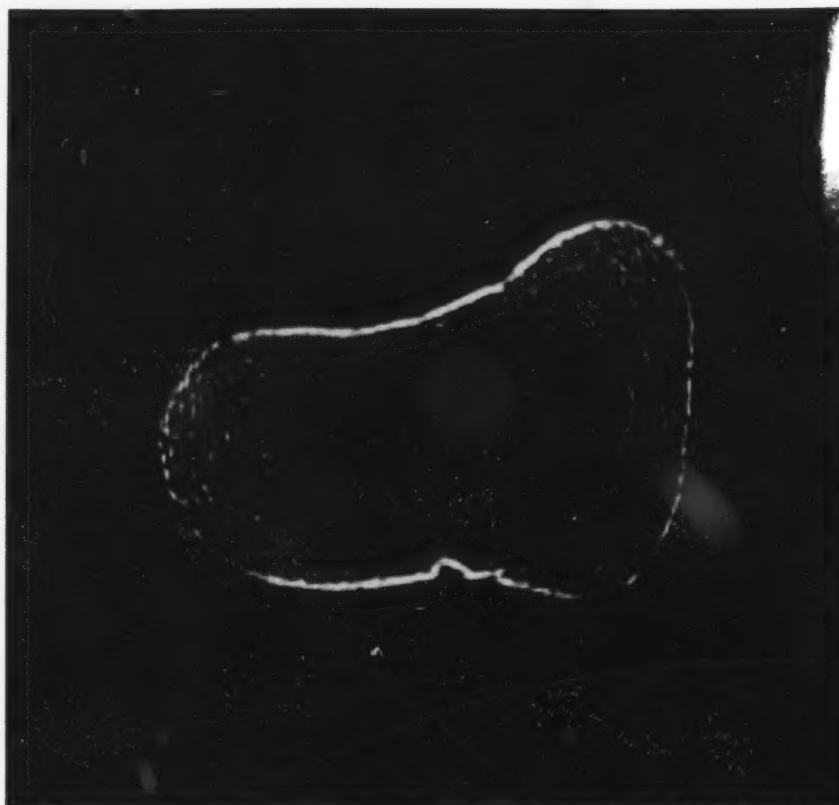


Figure 5.4.7: The restored image of the cadaver femur at a distance of 4.5 cm away from the distal end, using the Wiener filter.



Figure 5.4.8: The restored image of the cadaver femur at a distance of 2.5 cm away from the distal end, using the locally adaptive method in the Cartesian coordinates.



Figure 5.4.9: The restored image of the cadaver femur at a distance of 2.5 cm away from the distal end, using the Wiener filter.

in contrast to an iterative one for the locally adaptive and the polar coordinate techniques. In the case of the restoration performed in the regularly sampled polar coordinate system, the storage requirements (29.8 Kwords) for the pre-computed PSF are much smaller than for the locally adaptive method based on the quadrant symmetry approach (584 Kwords). However, the computational requirements for both methods are nearly the same because of the iterative procedures used in both cases.

It should also be mentioned that the fundamental limitations on the spatial resolution improvement achieved by image restoration are set by the spacing between samples in the projections, and by the noise characteristics. Both the sample spacing and the pixel size for the scanner used is 0.34 mm at the high resolution setting. Noise is a serious problem for this scanner since the X-ray tube is operated at 50 KV and 2.8 mA, for ultra low-dose scanning (0.03 mSv per slice), and the images are less than ideal starting points for restoration. Modern commercial CT scanners, however, have reconstruction pixel size, sample spacing of projections (< 0.3 mm) and noise levels which are more favorable for achieving better spatial resolution via restoration than that obtained by using only the filtered backprojection method for image reconstruction. For example, a SOMATOM PLUS scanner achieves a high contrast spatial resolution of 0.5 mm for a 16 cm diameter phantom with a scan time of 1 sec (300 mA). However, the reconstruction pixel size of this commercial scanner ranges between 0.06 mm to 1 mm. The focal spot dimensions (0.7 by 0.9 mm), however, will limit the spatial resolution achievable despite the small pixel size and sample spacing of the projections. Therefore, a suitable image restoration procedure can be applied to the commercial CT scanner images to improve the spatial resolution limited by the focal spot size.

6. THREE-DIMENSIONAL IMAGE RESTORATION

To this point, both the X-ray focal spot intensity and detector sensitivity distributions were assumed to be 1-D functions defined along a line contained within the slice plane. The X-ray beam profiles resulting from this assumption were planar, 2-D functions in the (x,y) or slice plane, denoted as $K_0(x,y; r,\theta)$ since these functions were different at different beam positions (r,θ) . The data collection and image reconstruction procedures for a CT system could, therefore, be written as a 2-D object to 2-D image mapping where the mapping functions were given by the 2-D PSF. In an actual CT system, however, the X-ray focal spot intensity and detector sensitivity distributions are 2-D functions defined along a line in the slice plane (transaxial) and in the perpendicular direction to the slice plane (axial). Therefore, the X-ray beam profiles of a CT system should be considered over an elongated volume resulting from the 2-D X-ray focal spot and detector aperture. The resulting PSF of a CT scanner is a 3-D function since the 2-D images reconstructed from the projections collected at a fixed slice position are formed by both the in-plane and out-of-plane object distributions. Image restoration considered in the previous chapters improves the spatial resolution in the reconstructed 2-D images. However, a 2-D image does not accurately measure the attenuation properties of an object in the slice plane, even when no blurring occurs within the slice, since the out-of-plane contributions must be inverted using a 3-D image restoration. Due to this blurring in the axial direction, it is customary to obtain the images in the non-overlapping planes of an object since the images taken at the overlapping positions tend to provide similar anatomical and structural information. The size of the X-ray beam in the axial direction is usually 3 or more times larger than the size within the slice plane, which makes the slices 1 to several millimeters thick. In some applications, such as a 3-D display of the object from a set of images, a large number of finely spaced slices are required. A 3-D image restoration would be beneficial in such applications before any further processing of the images.

An ideal 3-D reproduction of the object distribution from a set of 2-D CT images can only be obtained using 3-D image restoration. The simplification of 3-D image restoration to 2-D image restoration is possible only if either of the following approximations is true.

- (i) The distribution of the attenuation properties of the object is approximately symmetric in the axial direction.

(ii) The 3-D PSF of a CT scanner is separable into a 2-D PSF in the (x,y) plane which is the same as in the previous chapters, and a 1-D PSF in the axial direction.

The attenuation properties of the human body are seldom axially symmetric. In general, the 3-D PSF cannot be separated into 2-D and 1-D PSFs.

In the following section, a formulation of 3-D image restoration for a CT system is described. A number of approximations are suggested under which the 3-D PSF can be separated into a product of a 2-D PSF and a 1-D PSF in the axial direction.

6.1 The 3-D Formulation of the Image Restoration Problem

Let $f(x,y,z)$ be a 3-D, real valued function which denotes the distribution of attenuation properties of an object. In Figure 6.1.1, an X-ray beam is shown at a position when the CT system is collecting projections of the object at a slice position located at a distance b from the $z=0$ plane, in the z direction. Within the $z=b$ plane, the beam is shown rotated by an angle θ from x -axis, located at a distance r from the origin. Let $K_0(x,y,z; r,\theta,b)$ define the X-ray beam profile at this position. The measured projection $p_s(r,\theta,b)$ is related to the object distribution $f(x,y,z)$ via the following triple integral.

$$p_s(r,\theta,b) = \int \int \int f(x,y,z) K_0(x,y,z; r,\theta,b) dx dy dz \quad (6.1.1)$$

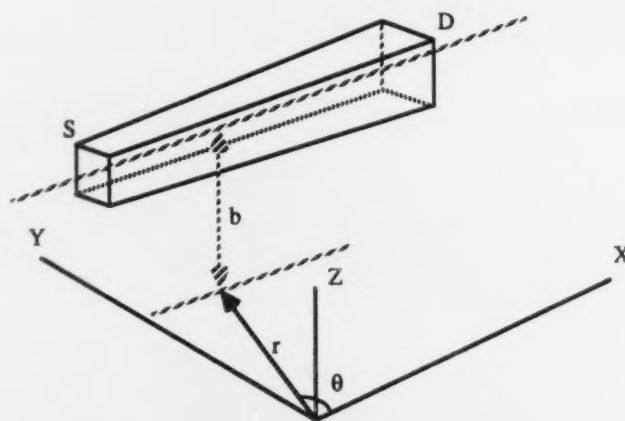


Figure 6.1.1: The (x,y,z) -axes associated with the object being imaged are such that the z -axis coincides with the axis of rotation of the moving gantry. The position of the beam center-line in that frame is determined by the three parameters: the axial position b , the angular position θ and the translational position r .

This linear relationship between the object distribution and its projections can be derived as given in section (3.1.1) for the case of a 2-D object. The derivation assumes that the line integrals within the beam volume do not change appreciably.

The filtered backprojection algorithm is used to reconstruct an image from the projections collected in the $z=b$ plane. The 3-D PSF of a CT system can be derived by assuming that the object distribution is a single point located at (x_1, y_1, z_1) . The projections of this point-object, at all positions (r, θ, z) , are given by the beam profiles $K_0(x_1, y_1, z_1; r, \theta, z)$. Therefore, the 3-D PSF $h(x, y, z; x_1, y_1, z_1)$ can be written as

$$h(x, y, z; x_1, y_1, z_1) = \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} K_0(x_1, y_1, z_1; r, \theta, z) h_s(x \cos \theta + y \sin \theta - r) dr d\theta \quad (6.1.2)$$

where $h_s(r)$ is the impulse response of the filter used in the reconstruction procedure.

The reconstructed images $g(x, y, z)$ at each of the slice positions ($z = \text{constant}$) can be related to the object distribution $f(x, y, z)$ as follows.

$$g(x, y, z) = \int \int \int f(x_1, y_1, z_1) h(x, y, z; x_1, y_1, z_1) dx_1 dy_1 dz_1 \quad (6.1.3)$$

Therefore, the 3-D image restoration problem is the same as inverting the triple superposition integral of equation (6.1.3).

The PSF width in the z direction is determined by the width of the X-ray beam profiles in that direction. In practice, a CT system collects projections at a fixed position in the z direction. The gantry system containing the X-ray source and the detector collimator then moves to another position to collect the projection data for another image. Therefore, the function $g(x, y, z)$ is sampled in all three directions. Equation (6.1.3) makes the contribution of out-of-plane object distribution obvious since an image at a specified z position contains weighted contributions from the object distribution at a number of z positions. If the slices obtained by a CT system are more widely spaced than the PSF width in the z direction, then no improvement can be obtained by 3-D image restoration since consecutive slices are independent of each other. The slices taken by a CT system must be finely spaced to obtain the improved 3-D representation of object distribution using image restoration. The required number of slices per unit distance can be determined by considering the PSF in the z direction.

For the complete formulation of 3-D image restoration, a noise term must be added to the right-hand-side of equation (6.1.3). This term is ignored in the restoration method described in the next section since the main purpose is to demonstrate the complexity of 3-D image restoration procedure.

The solution of the discretized form of equation (6.1.3) can be carried out using the storage-based implementation in the Cartesian coordinate system. In section 5.2.1.3, the storage requirement for the spatially variant PSF were determined to be 0.584 megawords for a 257-by-257 pixel image and a 9-by-9 pixel 2-D PSF. In the case of 3-D image restoration, the storage requirement increases according to the sampling in the z direction. If the PSF in the z direction contains twice the number of samples in the x or y direction, the storage for a $9 \times 9 \times 19$ pixel PSF increases to about 11 megawords. The computational requirements also increase both due to the size of the PSF in the z direction, and the number of slices to be used in the restoration procedure.

6.1.1 Image Restoration by Coordinate Transformation

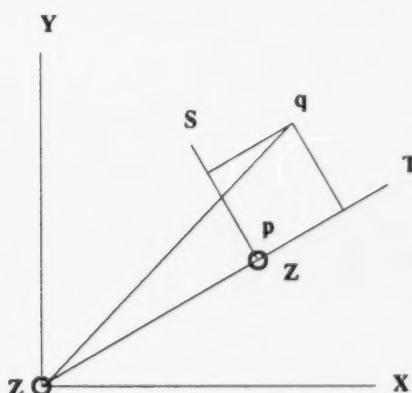


Figure 6.1.2: Transformation of the Cartesian coordinate system (x,y,z) to the cylindrical coordinate system (r,ϕ,z) via the rotating coordinate system (t,s,z) . The values of the three coordinate system, respectively, at point p are given by (x_1,y_1,z_1) , (r_1,ϕ_1,z_1) and $(0,0,z_1)$. The z axis is denoted by circles as if the coordinate systems are viewed from the top. q is an arbitrary point with coordinates (x,y,z) , (r,ϕ,z) and (t,s,z) .

The image restoration by coordinate transformation method was suggested by J.G.Verly [92] in which the spatial variations in the PSF are utilized to formulate the image restoration problem in cylindrical coordinates. In Figure 6.1.2, a point-object is assumed to be located at point p (x_1,y_1,z_1) which is denoted as (r_1,ϕ_1,z_1) in cylindrical coordinate system. Coordinate system (t,s,z) is the rotated Cartesian coordinate system whose origin is located at point p. It was shown in section 4.2.3 that the PSF for point-objects, located

within the (x,y) plane at a fixed radial distance r_1 from the origin, is spatially invariant in the (t,s,z) coordinates. Consider a point q(x,y,z) whose cylindrical coordinates are given by (r,θ,z). The (t,s) values for this point can be computed as follows.

$$t = r \cos(\theta - \theta_1) - r_1 \quad (6.1.4)$$

$$s = r \sin(\theta - \theta_1) \quad (6.1.5)$$

where r , and r_1 are the distances of points p and q, respectively, from the origin of the (x,y,z) coordinates. Therefore, if the images $g(x,y,z)$ are transformed to $g(r,\theta,z)$ in cylindrical coordinates, the 3-D PSF becomes independent of the angular position of the point-object. The X-ray beam profiles remain identical from one z position to another making the PSF independent of the z direction. Equation (6.1.3) can thus be written in cylindrical coordinates as

$$g(r,\theta,z) = \int \int \int f(r_1, \theta_1, z_1) h\{r \cos(\theta - \theta_1) - r_1, r \sin(\theta - \theta_1), z - z_1; r_1\} r_1 dr_1 d\theta_1 dz \quad (6.1.6)$$

After putting $h^+(r, \theta - \theta_1, z - z_1; r_1) = h\{r \cos(\theta - \theta_1) - r_1, r \sin(\theta - \theta_1), z - z_1; r_1\} r_1$, equation (6.1.6) can be written as the following.

$$g(r,\theta,z) = \int \int \int f(r_1, \theta_1, z_1) h^+(r, \theta - \theta_1, z - z_1; r_1) dr_1 d\theta_1 dz \quad (6.1.7)$$

For fixed values of r and r_1 , equation (6.1.7) is the same as a 2-D convolution of the object distribution and the PSF in the θ and z directions. Let $F(r_1, \Omega_\theta, \Omega_z)$ be 2-D Fourier transform of the object distribution at a fixed radial position r_1 . Similarly, let $H^+(r, \Omega_\theta, \Omega_z; r_1)$ be the 2-D Fourier transform of the 3-D PSF since the PSF is spatially invariant at the fixed radial position. It should be noted that the Fourier transform in the θ variable is discrete since the spatial domain functions are periodic with a period of 2π . This 2-D Fourier transform can be computed in two steps. In the first step, the conventional 1-D Fourier transform in the z direction is calculated at all θ positions. A 1-D cyclic form of the transform is then computed at each spatial frequency Ω_z . Using the convolution theorem [93], equation (6.1.7) can be written as:

$$G(r, \Omega_\theta, \Omega_z) = \int_0^\infty F(r_1, \Omega_\theta, \Omega_z) H^+(r, \Omega_\theta, \Omega_z; r_1) dr_1 \quad (6.1.8)$$

where $G(r, \Omega_\theta, \Omega_z)$ is the 2-D Fourier transform of the reconstructed image in the θ and z directions.

The case when each 2-D reconstructed image is sampled at N and M discrete radial and angular positions is considered in the following discussion. In addition, it is assumed that there are a total of L transaxial slices available for 3-D image restoration. Therefore, $G(r, \Omega_\theta, \Omega_z)$ can be calculated at M cyclic and L spatial frequencies in the θ and z directions, respectively. The integral in equation (6.3.8) can be approximated as the summation as:

$$G(r_i, \Omega_j, \Omega_k) = \Delta r \sum_{n=1}^N H^+(r_i, \Omega_j, \Omega_k; r_{1,n}) F(r_{1,n}, \Omega_j, \Omega_k) \quad (6.1.9)$$

where Δr is the radial sample spacing, and $r_n, r_{1,k}, \Omega_j$ and Ω_k are discrete variables corresponding to r, r_1, Ω_θ , and Ω_z . The indices i, j and k are in the range 1 to N , 1 to M and 1 to L , respectively. The 2-D Fourier transforms of the set of L reconstructed images can be computed by using M, L -point and L, M -point FFTs, first in the z directions for each angular position and then in the θ direction at each spatial frequency Ω_k . Equation (6.3.9) can be written in the matrix form for Fourier coefficient (j,k) as:

$$G_{jk} = H_{jk} F_{jk} \quad (6.1.10)$$

where

$$H_{jk} = \begin{bmatrix} H^+(r_1, \Omega_j, \Omega_k; r_{1,1}) & \dots & H^+(r_1, \Omega_j, \Omega_k; r_{1,N}) \\ \dots & \dots & \dots \\ H^+(r_N, \Omega_j, \Omega_k; r_{1,1}) & \dots & H^+(r_N, \Omega_j, \Omega_k; r_{1,N}) \end{bmatrix}$$

and

$$G_{jk} = \begin{bmatrix} G(r_1, \Omega_j, \Omega_k) \\ \dots \\ G(r_N, \Omega_j, \Omega_k) \end{bmatrix}$$

and

$$F_{jk} = \begin{bmatrix} F(r_{1,1}, \Omega_j, \Omega_k) \\ \dots \\ F(r_{1,N}, \Omega_j, \Omega_k) \end{bmatrix}$$

Equation (6.3.10) is a set of $(L \times M)$ equations corresponding to each of the $(L \times M)$ Fourier coefficients. For each coefficient, in order to determine the solution F^*_{jk} , an N -by- N size matrix must be inverted. A direct matrix inversion may not be possible since the matrix H_{jk} is likely to be ill-conditioned. A pseudo-inverse or a regularized minimum mean squared solution which takes the noise characteristics into account can be determined. If it is assumed that the pseudo-inverse is given by H^*_{jk} , then the solution of equation (6.3.10) can be written as:

$$F^*_{jk} = H^*_{jk} G_{jk} \quad (6.1.11)$$

The 3-D image restoration procedure can now be summarized as follows. In the first step, 2-D Fourier transforms of the set of reconstructed images, and the 3-D PSF, in the θ and z directions are calculated at each of N discrete radial positions. For each of the $(L \times M)$ Fourier coefficients, a pseudo-inverse of matrix H_{jk} is then used in equation (6.3.11) to provide the solution vector F^*_{jk} . These N -point solution vectors are then re-ordered into 2-D sequences at each of the radial position, and the inverse Fourier transform in the θ and z directions is used to obtain the set of restored images in the cylindrical coordinate system.

The storage requirement for the entire 3-D image restoration procedure can be estimated assuming that the pseudo-inverse matrices are pre-computed and stored. If $N=129$, $M=360$ and $L=128$, then each 257-by-257 image in a set of 128 2-D images is transformed into 129 radial and 360 angular samples. Therefore, a pseudo-inverse matrix for each of the $(L \times M)$ coefficients will be of dimensions 129-by-129. The total storage requirement will then be $(360 \times 128 \times 129 \times 129) 0.767$ gigawords. This can be reduced by storing only the non-zero elements of the pseudo-inverse matrices. However, if the noise correlation matrix is to be used to compute the regularized inverse matrices then it is unlikely that many elements of these matrices will be zero.

In conclusion, 3-D restoration in cylindrical coordinates is possible when sufficient memory is available with the computer. The idea of providing the storage-based method is to devise a computationally efficient restoration. The computational requirements are summarized as follows:

- (1) $2 \times M$ number of L -point FFTs, and $2 \times L$ number of M -point FFTs;
- (2) a total of $(L \times M)$ number of multiplications of a N by N matrix with an N -point vector.

However, for an adaptive restoration procedure the pseudo-inverses must be computed for each of the objects scanned where the computational considerations have to take into account the number of multiplications required to compute a total of $(L \times M)$ inverse matrices each of dimension N -by- N . It should also be noted that the computational requirements will be significantly higher if an image restoration procedure is to be implemented in the Cartesian coordinate system. Therefore, it is very unlikely that a 3-D image restoration procedure can be implemented on a sequential computer.

6.1.2 Decomposition of 3-D Restoration into 2-D Restoration

From the previous discussion, it would seem that ideal 3-D image restoration is not practical at this time. The coordinate transformation either requires a large storage area for the pre-computed pseudo-inverse matrices or a large computational burden is associated in computing the pseudo-inverse matrices. Although, no analysis of the computational requirements is provided for the Cartesian coordinate implementation, this would be much higher than the cylindrical coordinates since FFTs cannot be utilized, and an iterative procedure for inverting the matrix would be required.

The practical procedure for producing 3-D restored images is to decompose the 3-D PSF into a product of 2-D and 1-D PSFs. The X-ray beam profiles are spatially invariant in the z direction thereby allowing the notation to be changed from $K_0(x,y,z; r,\theta,z)$ to $K_0(x,y,z; r,\theta)$. The first step in studying the physical constraints which make the 3-D PSF separable into 2-D and 1-D PSFs, is to determine whether the 3-D X-ray beam profiles are separable as:

$$K_0(x,y,z; r,\theta) = K_{0,1}(x,y; r,\theta) K_{0,2}(z,y; r,\theta) \quad (6.1.12)$$

Since the X-ray beam profiles are calculated as a convolution of the 2-D X-ray focal spot intensity and detector sensitivity distributions (section 4.2.1), a decomposition of the beam profiles is only possible if these distributions are decomposable themselves. The detector sensitivity distribution, being uniform over the rectangular aperture, can be separated into a product of 1-D functions. Moreover, in a study of X-ray focal spots, Burgess [94] showed that, in most cases, one introduces very little error by assuming that the 2-D source intensity distribution is separable.

After substituting equation (6.1.12) into (6.1.2), the expression for the 3-D PSF can be modified to:

$$h(x,y,z; x_1,y_1,z_1) = \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} K_{0,1}(x_1,y_1; r,\theta) K_{0,2}(z-z_1,y_1; r,\theta) h_s(x\cos\theta+y\sin\theta-r) dr d\theta \quad (6.1.13)$$

The X-ray beam profile in (x,y) plane, $K_{0,1}(x_1,y_1; r,\theta)$ varies as a function of the position (r,θ) since each of the r positions is determined by a different detector. Each detector sees a different focal spot as shown in section 4.2.2. The X-ray beam profile in the (z,y) plane, $K_{0,2}(z-z_1,y_1; r,\theta)$, however, is independent of the position r since the apparent focal spot for all the detectors in the axial direction is the same. Therefore, one can write $K_{0,2}(z-z_1,y_1; r,\theta)$ as $K_{0,2}(z-z_1,y_1;\theta)$. The dependence of function $K_{0,2}(z-z_1,y_1;\theta)$ on the angular position is possible since the beam in the z direction diverges from the focal spot to the detector aperture. This can be seen by visualizing the rotating beam in terms of the rotating (t,s) coordinates since the function in z direction would be dependent on the value of s. However, if one assumes that the size of the object being scanned is very small compared to the distance between the X-ray focal spot and detector aperture, then the divergence of the beam, as a function of s in (t,s) coordinates, can be ignored. The s value is calculated as a function of y_1 and θ in equation (6.3.5) since θ_1 is determined by y_1 . Therefore, if the beam profile is assumed to be non-divergent in s, then the function $K_{0,2}(z-z_1,y_1;\theta)$ can be assumed independent of both y_1 and θ , as a function $K_{0,2}(z-z_1)$. Under this assumption, the expression for the 3-D PSF can be modified to the following.

$$h(x,y,z; x_1,y_1,z_1) = K_{0,2}(z-z_1) \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} K_{0,1}(x_1,y_1; r,\theta) h_s(x\cos\theta+y\sin\theta-r) dr d\theta \quad (6.1.14)$$

The function inside the double integral can be realized as the 2-D PSF $h(x,y; x_1,y_1)$ as described in previous chapters of this thesis. Hence, the 3-D PSF can be written as a product of 2-D spatially variant PSF in the (x,y) plane, and a 1-D spatially invariant PSF in the z direction.

The possibility of separating the 3-D PSF should be explored in a different way even if the approximation made in the previous discussion is violated in practice. This can be done by computing the 3-D PSFs at various points in the scan field and then fitting a separable composite 3-D function to this PSF, similar to the 2-D PSF parametrization of section 4.2.3. If a separable function is able to account for the most of the energy in the 3-D PSF, a restoration procedure described in the following paragraphs could be applied.

For a separable PSF, the input-output relationship can be modified from equation (6.1.3).

$$g(x,y,z) = \int K_{0,2}(z-z_1) \left\{ \int \int f(x_1,y_1,z_1) h(x,y; x_1,y_1) dx_1 dy_1 \right\} dz_1 \quad (6.1.15)$$

This equation can be modified into a series of 2-D image restorations and a number of 1-D restorations in the z direction.

$$g(x,y,z) = \int K_{0,2}(z-z_1) g'(x,y,z_1) dz_1 \quad (6.1.16)$$

$$g'(x,y,z_1) = \int \int f(x_1,y_1,z_1) h(x,y; x_1,y_1) dx_1 dy_1 \quad (6.1.17)$$

In the first step, a spatially invariant 1-D restoration, in the z direction, can be carried out from equation (6.3.15) at each (x,y) position in the set of reconstructed images $g(x,y,z)$. A Wiener filter restoration is possible and the noise in the consecutive images can be assumed to be uncorrelated. For a 257-by-257 image in the (x,y) plane, this would require nearly 66000 Wiener filtering operations which could be done very fast without any significant storage requirements. As a result of the Wiener restoration, one would obtain the set of images $g'(x,y,z_1)$. Each image in this set can be restored utilizing the storage based implementation of the 2-D image restoration procedure. If the 2-D image restoration is regularized using the noise properties, the noise variance in the 1-D restored set is needed which can be easily computed since the Wiener filter restoration is a linear operation. In any case, the storage the spatially variant PSF in one quadrant of the (x,y) plane is required, and the computational complexity for each 2-D image restoration procedure is the same as given in section 5.2.

7. SUMMARY AND CONCLUSIONS

7.1 SUMMARY

Restoration of CT images using a spatially variant implementation of the locally adaptive method in Cartesian coordinates is superior to both the same restoration method implemented in polar coordinates and the Wiener filter restoration which assumes the PSF to be spatially invariant. The polar coordinate restoration procedure is not able to take spatially correlated noise into account since the spatially invariant autocorrelation function in the Cartesian coordinate system becomes spatially variant in the polar coordinate system. The interpolation required to transform the image from one coordinate system to the other further degrades the visual quality of the restored images. The Wiener filter method results in variable spatial resolution in the restored image because the spatially invariant PSF approximation becomes poor when an object occupies a large area of the scan field.

The spatially variant locally adaptive method can therefore be offered as an option, in addition to the filtered backprojection method, with commercial CT systems. Measurements of the PSF and the NPS for a particular CT system are required for the implementation of this restoration procedure. In addition, the object boundary and the noise variance in each pixel of the reconstructed image are required. Incorporating the boundary constraint is useful, both in view of the ill-conditioned nature of image restoration problem and computational requirements, since pixel values outside the boundary need not be processed. The object boundary is estimated directly from the projection data.

Although, the PSF of a commercial CT system can be measured by scanning a water phantom containing thin wires located at various points, the measured PSF contains errors due to noise and to the off pixel grid location of the wires. It should be remembered that image restoration theoretically involves inverting a blur matrix formed by the PSF. If the measured PSF contains errors, so does the inverted matrix and errors in the restored images result. In chapter 4, an estimation of the PSF using known X-ray beam profiles was discussed. X-ray beam profiles were modeled as a convolution of the focal spot intensity and detector sensitivity distributions. For the special purpose CT scanner, an experimental procedure provided focal spot intensity distribution measurements with a coefficient of variation of 2%. The experimental procedure utilized for measuring the focal spot intensity distribution is applicable only to the special purpose CT system available to this project since commercial CT systems are incapable of horizontal translation of the

X-ray tube and detector assembly. There are other methods for estimating the focal spot intensity distribution of an X-ray tube, such as imaging the focal spot using a pin-hole camera, and estimating the focal spot from the images of a star pattern [90]. For an accurate estimation of the PSF for a CT system, detector angles and anode heel angle in the X-ray tube must be known so that the data collection procedure can be correctly simulated.

There are several techniques for accurately estimating the NPS in reconstructed CT images. The NPS should be averaged over a large number of water phantom images in order to reduce the variance of the estimate. If the measured NPS is circularly symmetric a further reduction in the variance can be obtained by averaging radial profiles. A circularly symmetric NPS results if bandlimited interpolation is used before the backprojection.

The locally adaptive image restoration method has the disadvantage that it assumes the noise in the reconstructed image to be stationary to determine the pre-whitening filter. An ideal image restoration method would pre-whiten the spatially correlated and non-stationary noise by inverting the noise autocorrelation matrix. A procedure for determining the spatially variant autocorrelation function has been reported [91]. However, an ideal restoration method would be very complex since the spatially variant noise autocorrelation function must be estimated for each image. Moreover, the resulting autocorrelation matrix increases the storage requirement. These difficulties, along with the fact that the spatial variations in the autocorrelation function within the scan field are small except at the edge of the scan field [91], suggest that a stationary approximation for the noise is reasonable.

The main drawback of the locally adaptive, spatially variant restoration procedure, compared to the filtered backprojection method, is the computational load. This is because the algorithm contains a double iteration; one for the conjugate gradient method to solve the normal equations and the other for determining the optimal value of the Lagrange multiplier γ . In chapter 3, the analysis of the NSE between the original and the restored images as a function of this multiplier demonstrated a range of values for which the %NSE was minimum (Figures 3.2.4 and 3.2.5). Although, the criterion for selecting the multiplier (equation 3.1.23) always produced the value within this range, this is not known *a priori*. Therefore, the iterations are needed not only to solve the system of equations resulting from the locally adaptive formulation for each multiplier value, but the entire procedure must be repeated for a number of values in order to satisfy the constraint 3.1.23. The number of iterations can, however, be reduced or completely avoided if only a single class of images are to be restored. This is due to the fact that the optimal value of the multiplier does not

change significantly from one image to another. Even if the iteration over the multiplier is avoided, the conjugate gradient method is computationally very slow. To restore an image may take 6 to 8 minutes on a SUN SPARC+2 workstation. Therefore, further work is required in order to make the restoration algorithm more efficient.

Further work is also required to determine whether it is possible to decompose the 3-D restoration of CT images to 2-D and 1-D restorations. For this purpose, experimental measurements of the focal spot intensity and detector sensitivity distributions in the direction perpendicular to the slice plane are required. The 3-D restored images can then be obtained if the X-ray beam profile, estimated from these measurements, is found to be approximately non-divergent in the direction perpendicular to the slice plane.

7.2 CONCLUSIONS

It is concluded that spatial resolution in CT images can be improved by using 2-D image restoration; the restoration procedure can be implemented using either the projection data or the reconstructed images. The locally adaptive image restoration method provides the smallest %NSE between the original and the restored images. This method also resulted in the largest area under the ROC curves suggesting the best discrimination in a hypothetical binary classification scheme.

The estimation of the spatially variant PSF of a CT scanner by using known X-ray beam profiles results in more accurate PSFs than those measured by scanning phantoms. Thus, this present work also provides an improved procedure for estimating the spatially variant PSF of commercial CT scanners.

The best results for image restoration applied to CT images are obtained by using storage-based implementation of the locally adaptive method in Cartesian coordinates. The quadrant-symmetry-based storage scheme results in the best compromise between storage and computational requirements.

8. REFERENCES

- [1] J. Radon, "Ueber die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten," *Ber Verh Saechs Akad Wiss*, vol. 69, pp. 262-277, 1917.
- [2] R. W. Bracewell, "Strip integration in radio astronomy," *Aus J Phys*, vol.9, pp. 198-217, 1956.
- [3] A. M. Cormack, "Representation of functions by its line integrals with some radiological applications-I," *J Appl Phys*, vol.34, pp. 2722-2727, 1963.
- [4] A. M. Cormack, "Representation of functions by its line integrals with some radiological applications-II," *J Appl Phys*, vol. 35, pp. 2908-2913, 1964.
- [5] G. N. Hounsfield, "Computerized transverse axial scanning (): Part 1. Description of system," *Br J Radiol*, vol.46, pp. 1016-1022, 1973.
- [6] E. C. McCullough *Radiology of Skull and Brain: Technical Aspects of Computed Tomography* (eds. T. H. Newton and D. G. Potts). London: The C.V. Mosby Company, 1981; pp 4301-4311.
- [7] W. H. Berninger, R. W. Redington, *Radiology of Skull and Brain: Technical Aspects of Computed Tomography* (eds. T. H. Newton and D. G. Potts). London: The C.V. Mosby Company, 1981; pp 4261-4285.
- [8] T. Greitz and M. Bergström, *Radiology of Skull and Brain: Technical Aspects of Computed Tomography* (eds. T. H. Newton and D. G. Potts). London: The C.V. Mosby Company, 1981; pp 4286-4295.
- [9] P. Rüeggsegger, U. Elsässer, and M. Anliker et al, "Quantification of bone mineralization using computed tomography," *Radiology*, vol. 121, pp. 93-97, 1976.
- [10] E. P. Durand, "Quantitative analyse von knochenstrukturen aus computertomogrammen von Gelenknahen Skelettabschnitten," *Ph.D. Dissertation, Eidgenössischen Technischem Hochschule, Zürich*, 1989.
- [11] R. A. Brooks and G. D. Chiro, " Statistical limitations in x-ray reconstructive

- tomography," *Med. Phys.*, vol. 3, no 4, pp. 237-240, 1976.
- [12] L. A. Shepp and B. F. Logan, "The Fourier reconstruction of a head section," *IEEE Trans. Nucl. Sci.*, vol. NS-21, pp. 21-43, June 1974.
 - [13] Gabor T. Herman, *IMAGE RECONSTRUCTION FROM PROJECTIONS: The fundamentals of Computerized Tomography*. New York: Academic Press, 1980.
 - [14] T. F. Budinger, G. T. Gullberg, and R. H. Huesman, "Emission Computed Tomography," in *Image Reconstruction from Projections: Implementation and Applications*. (G. T. Herman, ed.) Berlin and New York: Springer-Verlag, 1979, pp. 147-246.
 - [15] R. A. Crowther, D. J. DeRosier, and A. Klug, "The reconstruction of a three-dimensional structure from projections and its application to electron microscopy," *Proc. R. Soc. London*, vol A317, pp. 319-340, 1970.
 - [16] R. N. Bracewell and A. C. Riddle, "Inversion of fan beam scans in radio astronomy," *Astrophys. J.*, vol. 150, pp. 427-434, 1967.
 - [17] R. Gordon, "A tutorial on ART (Algebraic Reconstruction Techniques)," *IEEE Trans. Nucl. Sci.*, vol. NS-21, pp. 78-93, 1974.
 - [18] R. Gordon, R. Bender, and G. T. Herman, "Algebraic Reconstruction Techniques (ART) for three-dimensional electron microscopy and x-ray photography," *J. Theor. Biol.*, vol. 29, pp. 471-481, 1970.
 - [19] G. T. Herman and S. W. Rowland, "Three methods for reconstructing objects from x-rays: a comparative study," *Comput. Graphics Image Processing*, vol. 2, pp. 151-178, 1973.
 - [20] H. Stark et al., "Direct Fourier reconstruction in computer tomography," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-29, no. 2, pp. 237-245, 1981.
 - [21] J. D. O'Sullivan, "A fast sinc function gridding algorithm for Fourier inversion in computer tomography," *IEEE Trans. Med. Imag.*, vol. MI-4, no. 4, pp. 200-207, 1985.

- [22] W. E. Higgins and D. C. Munson, Jr., " A Hankel transform approach to tomographic image reconstruction," *IEEE Trans. Med. Imag.*, vol. 7, no. 1, pp. 59-72, 1988.
- [23] G. N. Ramachandran and A. V. Lakshminarayanan, " Three dimensional reconstruction from radiographs and electron micrographs: application of convolutions instead of Fourier transforms," *Proc. Nat. Acad. Sci. U. S.*, vol. 68, pp. 2236-2240, 1971.
- [24] A. P. Dhawan, R. M. Rangayyan, and R. Gordon, " Image restoration by Wiener deconvolution in limited-view computed tomography," *Applied Optics*, vol. 24, no. 23, pp. 4013-4020, 1985.
- [25] B. R. Frieden and C. K. Zoltani, " Maximum bounded entropy: application to tomographic reconstruction," *Applied Optics*, vol. 24, no. 23, pp. 3993-3999, 1985.
- [26] K. H. Park and S. B. Park, "Maximum entropy image reconstruction for an object with opaque obstructions," *IEEE Trans. Med. Imag.*, vol. MI-6, no. 4, pp. 308-312, 1987.
- [27] A. Lent, " A convergent algorithm for maximum entropy image restoration, with a medical X-ray application, " in *SPSE Conference Proceedings* (R. Shaw, ed.). Washington D. C. : Society of Photographic Scientists and Engineers, 1977, pp. 249-257.
- [28] G. Minerbo, " MENT: A maximum entropy algorithm for reconstructing a source from projection data," *Computer Graphics and Image Processing*, vol. 10, pp. 48-68, 1979.
- [29] A. Mohammad-Djafari and G. Demoment, " Maximum entropy image reconstruction in X-ray and diffraction tomography," *IEEE Trans. Med. Imag.*, vol. 7, no. 4, pp. 345-354, 1988.
- [30] S. F. Gull and T. J. Newton, " Maximum entropy tomography," *Applied Optics*, vol. 25, no. 1, pp. 156-160, 1986.
- [31] R. T. Smith and C. K. Zoltani, " An application of the finite element method to maximum entropy tomographic image reconstruction," *J. Sci. Comp.*, vol. 2, no.

3, pp. 283-295, 1987.

- [32] S. Kawata and O. Nalcioglu, "Constrained iterative reconstruction by the conjugate gradient method," *IEEE Trans. Med. Imag.*, vol. MI-4, no. 2, pp. 65-71, 1985.
- [33] M. I. Sezan and H. Stark, "Image restoration by the method of convex projections: Part 2- Applications," *IEEE Trans. Med. Imag.*, vol. MI-1, pp. 95-101, 1982.
- [34] K. M. Hansen, "POPART: Performance optimized algebraic reconstruction technique," *Proc. SPIE*, vol. 1001, pp. 318-325, 1988.
- [35] R. A. Brooks and G. DiChiro, "Principles of computer assisted tomography (CAT) in radiographic and radioisotopic imaging," *Phys. Med. Biol.*, vol. 21, pp. 689-732, 1976.
- [36] A. Rosenfeld and A. C. Kak, *Digital Picture Processing*. New York: Academic press, 1982, pp. 353-430.
- [37] G. T. Herman, A. V. Lakshminarayanan, and A. Naparstek, "Convolution reconstruction technique for divergent beams," *Comput. Biol. Med.*, vol. 6, pp. 259-271, 1976.
- [38] C. R. Crawford, "CT filtration aliasing artifacts," *IEEE Trans. Med. Imag.*, vol. 10, no. 1, pp. 99-102, 1990.
- [39] J. Jelinek and T. R. Overton, "Reordering schemes for multiple-rotation fan-beam CT scanner," *IEEE Trans. Med. Imag.*, vol. MI-4, no. 4, pp. 215-221, 1985.
- [40] T. M. Peters and R. M. Lewitt, "Computed tomography with fan beam geometry," *J. Comput. Assist. Tomogr.*, vol. 1, no. 4, pp. 429-436, 1977.
- [41] C. D. Stockham, "A simulation study of aliasing in computed tomography," *Radiology*, vol. 132, pp. 721-726, 1979.
- [42] P. M. Joseph, R. D. Spital, and C. D. Stockham, "The effects of sampling on CT images," *Computerized Tomography*, vol. 4, pp. 189-206, 1980.
- [43] P. M. Joseph and R. A. Schulz, "View sampling requirements in fan beam computed tomography," *Med. Phys.*, vol. 7, no. 6, pp. 692-702, 1980.

- [44] D. L. Snyder and J. R. Cox, Jr., *Reconstruction Tomography in Diagnostic Radiology and Nuclear Medicine*. (M. M. Ter-Pogossian, M. E. Phelps, G. L. Brownell, J. R. Cox, D. O. David, and R. G. Evens, eds.), Baltimore: University Park Press, 1977, pp. 3-31.
- [45] S. J. Glick, M. A. King, and B. C. Penny, "Characterization of the Modulation Transfer Function of discrete filtered backprojection," *IEEE Trans. Med. Imag.*, vol. 8, no. 2, pp. 203-213, 1989.
- [46] L. A. Shepp and J. B. Krushal, "Computerized Tomography: the new medical X-ray technology," *Am. Math. Mon.*, vol. 85, pp. 420-439, 1978.
- [47] M. F. Kijewski and P. F. Judy, "The noise power spectrum of CT images," *Phys. Med. Biol.*, vol. 32, no. 5, pp. 565-575, 1987.
- [48] B. F. Logan, "The uncertainty principle in reconstructing functions from projections," *Duke Math J.*, vol. 42, pp. 661-706, 1975.
- [49] D. C. Chesler, S. J. Riederer, and N. J. Pelc, "Noise due to photon counting statistics in computed X-ray tomography," *J. Comput. Assist. Tomogr.*, vol. 1, no. 1, pp. 64-74, 1977.
- [50] D. E. Dudgeon and R. M. Mersereau, *Multidimensional Digital Signal Processing*. New Jersey: Prentice-Hall, 1984, pp. 350-352.
- [51] H. C. Andrews and B. R. Hunt, *Digital Image Restoration*. New Jersey: Prentice-Hall, 1977, pp. 132-140 and 150-151.
- [52] R. C. Gonzalez and P. Wintz, *Digital Image Processing*. Menlo Park: Addison-Wesley Publishing Company, 1987, pp. 213-219.
- [53] J. Biemond, R. L. Lagendijk, and R. M. Mersereau, "Iterative methods for image deblurring," *Proceedings of the IEEE*, vol. 78, no. 5, pp. 856-883, 1990.
- [54] A. K. Katsaggelos and S. M. Efstratiadis, "A class of iterative signal restoration algorithms," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-38, no. 5, pp. 778-786, 1990.
- [55] N. B. Karayiannis and A. N. Venetsanopoulos, "Regularization theory in image

- restoration-The stabilization functional approach," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-38, no. 7, pp. 1155-1179, 1990.
- [56] D. M. Titterton, "General structure of regularization procedures in image reconstruction," *Astron. Astrophys.*, vol. 144, pp. 381-387, 1985.
 - [57] R. L. Lagendijk, J. Biemond, and D. E. Boeke, "Regularized iterative image restoration with ringing reduction," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-36, no. 12, pp. 1874-1888, 1988.
 - [58] E. T. Jaynes, "Prior probabilities," *IEEE Trans. System Science and Cybernetics.*, vol. SSC-4, no. 3, pp. 227-241, 1968.
 - [59] B. R. Frieden, "Restoring with maximum likelihood and maximum entropy," *J. Opt. Soc. Am.*, vol. 62, no. 4, pp. 511-518, 1972.
 - [60] X. Zhuang, E. Østevold, and R. M. Haralic, "A differential equation approach to maximum entropy reconstruction," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-35, no. 2, pp. 208-218, 1987.
 - [61] J. Skilling and S. F. Gull, "Algorithm and Applications," in *Maximum-Entropy and Bayesian methods in inverse problems* (eds. C. R. Smith and W. T. Grandy, Jr.). D. Reidel Publishing Company, 1985, pp. 83-132.
 - [62] M. I. Sezan and A. M. Tekalp, "Adaptive image restoration with artifact suppression using the theory of convex projections," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-38, no. 1, pp. 181-185, 1990.
 - [63] Y. Zhou, R. Chellappa, A. Vaid, and B. K. Jenkins, "Image restoration using a neural network," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-36, no. 7, pp. 1141-1151, 1988.
 - [64] R. L. Lagendijk, J. Biemond, and D. E. Boeke, "Identification and Restoration of noisy blurred images using the expectation-maximization algorithm," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-38, no. 7, pp. 1180-1191, 1990.
 - [65] A. A. Sawchuk and M. J. Peyrovian, "Restoration of astigmatism and curvature of field," *J. Opt. Soc. Am.*, vol. 65, no. 6, pp. 712-715, 1975.

- [66] B. C. Penny, M. A. King, and R. B. Schwinger, "Constrained least-squares restoration of nuclear medicine image: Selecting the coarseness function," *Med. Phys.*, vol. 14, no. 5, pp. 849-858, 1987.
- [67] K. Faulkner and B. M. Moores, "Analysis of x-ray computed tomography images using the noise power spectrum and autocorrelation function," *Phys. Med. Biol.*, vol. 29, no. 11, pp. 1343-1352, 1984.
- [68] K. M. Hanson, "Method of evaluating image-recovery algorithms based on task performance," *J. Opt. Soc. Am.*, vol. 7, no. 7, pp. 1294-1304, 1990.
- [69] B. S. Gottfried and J. Weisman, *Introduction to Optimization Theory*. Englewood Cliffs, New Jersey: Prentice-Hall, 1973, pp. 95-102.
- [70] A. Rosenfeld and A. C. Kak, *Digital Picture Processing vol. 3.1*. New York: Academic Press, 1982, pp. 390-397.
- [71] J. G. Verly and R. N. Bracewell, "Blurring in tomograms made with x-ray beams of finite width," *J. Comput. Assist. Tomogr.*, vol. 3, no. 5, pp. 662-678, 1979.
- [72] C. J. Bischof and J. C. Ehrhardt, "Modulation transfer function of the EMI CT head scanner," *Med. Phys.*, vol. 4, no. 2, pp. 163-167, 1977.
- [73] S. Doré, R. E. Kearney, and J. D. Guise, "Experimental determination of CT point spread function," *11th Annual International Conference of IEEE Engineering in Medicine & Biology Society*, pp. 620-621, 1989.
- [74] S. M. Bentzen, "Evaluation of the spatial resolution of a CT scanner by direct analysis of the edge response function," *Med. Phys.*, vol. 10, no. 5, pp. 579-581, 1982.
- [75] N. J. Schneiders and S. C. Bushong, "Single-step calculation of the MTF from ERF," *Med. Phys.*, vol. 5, no. 1, pp. 31-33, 1977.
- [76] R. T. Deoege and R. L. Morin, "A practical method to measure the MTF of CT scanners," *Med. Phys.*, vol. 9, no. 5, pp. 758-760, 1982.
- [77] S. Doré, R. E. Kearney, and J. D. Guise JD, "Quantitative assessment of CT PSF isotropicity and isoplanicity," *CMBEC-16-CCGB*, Winnipeg, pp. 31-32, 1990.

- [78] R. M. Gagne, "Geometrical aspects of computed tomography: Sensitivity profile and exposure profile," *Med. Phys.*, vol. 16, no. 1, pp. 29-37, 1988.
- [79] B. A. McClean, T. R. Overton, T. N. Hangartner, and S. Rathee, "A special purpose x-ray fan-beam CT scanner for trabecular bone density measurement in the appendicular skeleton," *Phys. Med. Biol.*, vol. 35, no. 1, pp. 11-19, 1990.
- [80] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes: The art of scientific computing (FORTRAN version)*. New York, Cambridge: Cambridge University Press, 1989, pp. 521-528.
- [81] P. M. Joseph and R. D. Spital, "The exponential edge gradient effect in x-ray computed tomography," *Phys. Med. Biol.*, vol. 26, no. 3, pp. 473-487, 1981.
- [82] G. T. Herman, "Correction for beam hardening in computed tomography," *Phys. Med. Biol.*, vol. 24, no. 1, pp. 81-106, 1979.
- [83] S. Rathee, "Quantification of trabecular bone density using single and dual energy x-ray computed tomography," *M.Sc. Thesis*, University of Alberta, Edmonton, Canada, pp. 47-57, 1988.
- [84] S. Doré, R. E. Kearney, and J. D. Guise, "Coordinate transformation for isoplastic representation of shift-variant CT point spread function," *Proceedings of 17th conference of the Canadian Medical and Biological Engineering Society*, pp. 169-170, 1991.
- [85] G. M. Robbins and T. S. Huang, "Inverse filtering for linear shift-variant imaging systems," *Proceedings of the IEEE*, vol. 60, no. 7, pp. 862-872, 1972.
- [86] A. A. Sawchuk, "Space-variant image motion degradation and restoration," *Proceedings of the IEEE*, vol. 60, no. 7, pp. 854-861, 1972.
- [87] A. A. Sawchuk, "Space-variant image restoration by coordinate transformations," *J. Opt. Soc. Am.*, vol. 64, no. 2, pp. 138-144, 1974.
- [88] K. R. Castleman, *Digital Image Processing*. Englewood Cliffs, New Jersey: Prentice-hall, 1979, pp 113-115.
- [89] J. Jelinek and T. R. Overton, "Simulation of CT reconstruction artifacts associated

with multiple-rotation fan-beam data collection," *Comput. Radiology*, vol. 10, no. 1, pp. 23-26, 1986.

- [90] H. E. Johns and J. R. Cunningham, *The Physics of Radiology*. Springfield: Charles C Thomas, 1983, pp. 591-596.
- [91] D. W. Wilson, B. M. W. Tsui, and J. A. Terry, " Non-stationary noise characterization for SPECT images," *IEEE Medical Imaging Conf.*, Santa Fe, New Mexico, pp. 66, 1991.
- [92] J. G. Verly, " High-resolution Imaging in 3-D reconstructive tomography," *Computer Graphics and Image Processing*, vol. 16, pp. 1-19, 1981.
- [93] R. N. Bracewell, *The Fourier Transform and Its Applications*, McGraw-Hill: New York, 1978.
- [94] A. E. Burgess, " Focal spots. I. MTF separability," *Invest. Radiol.*, vol 12., no. 1, pp. 36-43, 1977.

APPENDICES

I NOISE VARIANCE IN CT IMAGES

In this section, an expression of the noise variance in each pixel of the reconstructed image is derived. The derivation is provided by assuming that a two-step interpolation is used in filtered backprojection method.

Let $\sigma^2(na, m\Delta\theta)$ denote the position-dependent variance of noise $e(na, m\Delta\theta)$ in the projection at angle $m\Delta\theta$ containing L samples. For the two-step interpolation used in filtered backprojection method, the projection at each angle is filtered using a PSF with a $(2L-1)$ -samples long impulse response $h_s(na)$, and the number of samples in the filtered projections is doubled by convolving with an approximately bandlimited interpolating function $h_i(ka/2)$. For the purpose of this derivation, assume that both $e(na, m\Delta\theta)$ and $h_s(na)$ are inserted with zeros in between the known samples such that the sampling distance is the same as of the interpolating function $h_i(na/2)$. Let $h'_s(ka/2)$ denote the $(4L-3)$ -samples long impulse response of the filter function after inserting with zeroes and $h_d(ka/2)$ be the convolution between $h'_s(ka/2)$ and $h_i(ka/2)$.

$$h_d(ka/2) = \sum_{j=-P}^P h_i(ja/2)h'_s\{(k-j)a/2\} \quad \text{for } k=-(2L-2) \text{ to } (2L-2) \quad (\text{I.1})$$

where the size of the interpolating function is $2P+1$. Assuming that $e'(ka/2, m\Delta\theta)$ is the noise signal after inserting with zeroes, the noise signal $e_f(ka/2, m\Delta\theta)$ in the filtered projections containing twice the number of samples in the measured projections can be expressed as the following

$$e_f(ka/2, m\Delta\theta) = \sum_{j=0}^{2L-2} e'(ja/2, m\Delta\theta)h_d\{(k-j)a/2\} \quad (\text{I.2})$$

The noise variance $\sigma^2(ka/2, m\Delta\theta)$ is equal either to $\sigma^2(na, m\Delta\theta)$ or zero depending on whether k denotes a measured sample or a sample where a zero had been inserted.

The mean of the noise signal $e_f(ka/2, m\Delta\theta)$ is zero since the mean of $e'(ka/2, m\Delta\theta)$ is zero. Thus the variance $\sigma_f^2(ka/2, m\Delta\theta)$ of noise signal $e_f(ka/2, m\Delta\theta)$ can be computed as the following

$$\sigma_f^2(ka/2, m\Delta\theta) = E [e_f^2(ka/2, m\Delta\theta)] \quad (\text{I.3})$$

where E is an expectation operator.

Using equation (I.2), equation (I.3) can be written as the following

$$\sigma_f^2(ka/2, m\Delta\theta) = E \left[\sum_{j=0}^{2L-2} e'(ja/2, m\Delta\theta) h_d\{(k-j)a/2\} \sum_{i=0}^{2L-2} e'(ia/2, m\Delta\theta) h_d\{(k-i)a/2\} \right] \quad (I.4)$$

The order of summation and expectation can be interchanged such that

$$\sigma_f^2(ka/2, m\Delta\theta) = \sum_{j=0}^{2L-2} \sum_{i=0}^{2L-2} E [e'(ia/2, m\Delta\theta) e'(ja/2, m\Delta\theta)] h_d\{(k-i)a/2\} h_d\{(k-j)a/2\} \quad (I.5)$$

Since the noise in measured projections is spatially uncorrelated, the expectation term in equation (I.5) vanishes for all i and j such that $i \neq j$, and is equal to $\sigma^2(ja/2, m\Delta\theta)$ for $i=j$ so that

$$\sigma_f^2(ka/2, m\Delta\theta) = \sum_{j=0}^{2L-2} \sigma^2(ja/2, m\Delta\theta) h_d^2\{(k-j)a/2\} \quad (I.6)$$

The noise signal $v(ia, ja)$ in the reconstructed image is related to the noise signal in the filtered projections, e_f , as follows.

$$v(ia, ja) = \sum_{m=0}^{M-1} e_f\{ia \cos(m\Delta\theta) + ja \sin(m\Delta\theta), m\Delta\theta\} \quad (I.7)$$

where M is the number of projections. A linear interpolation of the filtered projections containing twice the number of samples in the measured projections is required since ' $ia \cos(m\Delta\theta) + ja \sin(m\Delta\theta)$ ' may not exactly coincide with a sample position $ka/2$. The noise variance at an arbitrary location ' $ia \cos(m\Delta\theta) + ja \sin(m\Delta\theta)$ ' can be estimated from the noise variances at sampled positions of the filtered projections. Assume that ' d ' is the sub-sample spacing between the location ' $ia \cos(m\Delta\theta) + ja \sin(m\Delta\theta)$ ' and the k^{th} sample position, a linear interpolation propagates the noise variance in the following manner.

$$\sigma_f^2[ia \cos(m\Delta\theta) + ja \sin(m\Delta\theta)] = (1-d)^2 \sigma_f^2(ka/2, m\Delta\theta) + d^2 \sigma_f^2[(k+1)a/2, m\Delta\theta] + 2d(1-d) E [e_f(ka/2, m\Delta\theta) e_f((k+1)a/2, m\Delta\theta)] \quad (I.8)$$

The term under the expectation is the autocorrelation function of the noise signal

$e_f(ka/2, m\Delta\theta)$ at one sample lag. The normalized autocorrelation function $R(1)$ is calculated as the autocorrelation function of $h_d(ka/2)$ at one sample lag since the autocorrelation function of noise signal $e'(ka/2, m\Delta\theta)$ is a δ -function.

$$R(1) = \frac{\sum_{k=-2L+2}^{2L-2} h_d(ka/2) h\{(k+1)a/2\}}{\sum_{k=-2L+2}^{2L-2} h_d^2(ka/2)} \quad (I.9)$$

The normalization constant is taken as the noise-variance at the k^{th} sample such that

$$E [n_f(ka/2, m\Delta\theta) n_f((k+1)a/2, m\Delta\theta)] = R(1) \sigma_f^2(ka/2, m\Delta\theta) \quad (I.10)$$

Since noise in the projections at different angles is spatially uncorrelated, the noise variance at pixel (i,j) is computed as

$$\sigma_v^2(ia, ja) = \sum_{m=0}^{M-1} \sigma_f^2[ia \cos(m\Delta\theta) + ja \sin(m\Delta\theta)] \quad (I.11)$$

The noise variance at each pixel in the reconstructed image can thus be calculated using equations (I.8) to (I.11).

II NOISE POWER SPECTRUM (NPS)

In this section, an expression of the NPS in CT images is derived with the assumption that noise in the projection data is stationary and spatially uncorrelated. The NPS of the continuous projections $S_{cp}(k)$ can thus be written as

$$S_{cp}(k) = \sigma^2 \quad (II.1)$$

where σ^2 is the noise variance in the measured projections and k denotes the spatial frequency. If the projections are sampled with a sampling distance 'a', then the NPS of the sampled projection is given by

$$S_{sp}(k) = \frac{\sigma^2}{L} \sum_{i=-\infty}^{\infty} \text{rect}(k-i/a) \quad (II.2)$$

where L is the number of samples in a projection, and $\text{rect}()$ is defined as

$$\text{rect}(x) = \begin{cases} 1 & \text{if } |x| < 1/2 \\ 0 & \text{otherwise} \end{cases} \quad (\text{II.3})$$

The sampled projection data is filtered using the impulse response $h_s(na)$. The Fourier transform $H_s(k)$ of this filter is expressed as a convolution between the ideal filter in frequency domain and the Fourier transform of a truncating window. A rectangular truncating window with half width equal to 'T' was used so that

$$H_s(k) = |k| * \frac{\sin(\pi k T)}{\pi k T} \quad (\text{II.4})$$

The NPS of filtered projections $S_{fsp}(k)$ can then be expressed as

$$S_{fsp}(k) = \frac{\sigma^2}{L} \sum_{i=-\infty}^{\infty} \text{rect}(k-i/a) |H_s(k-i/a)|^2 \quad (\text{II.5})$$

The NPS of the filtered projections containing twice the number of samples in the measured projections is given by

$$S_{dfsp}(k) = \frac{\sigma^2}{L} \sum_{i=-\infty}^{\infty} \text{rect}(k-2i/a) |H_s(k-2i/a)|^2 |H_i(k-2i/a)|^2 \quad (\text{II.6})$$

where H_i is the Fourier transform of an approximately bandlimited interpolating function.

A linear interpolation is used to determine the samples at arbitrary locations ($ia \cos \theta + ja \sin \theta$) from the values known at the sampling intervals of 'a/2'. Interpolation is the same as a convolution between the linear interpolating function and the filtered projections in spatial domain. In addition, the interpolated filtered projections are continuous. Therefore, if $G(k)$ is the linear interpolating function in the Fourier domain, then the NPS in the linearly interpolated filtered projections is expressed as

$$S_{dfsp}(k) = \frac{\sigma^2}{L} |G(k)|^2 \sum_{i=-\infty}^{\infty} \text{rect}(k-2i/a) |H_s(k-2i/a)|^2 |H_i(k-2i/a)|^2 \quad (\text{II.6})$$

The linearly interpolated projections are backprojected to reconstruct the image. According to the well known projection-slice-theorem, backprojection at a given angle adds the Fourier transform of the filtered projection to the 2-D Fourier transform of the reconstructed image along a spoke through the origin at that angle. The 2-D NPS $S(k_x, k_y)$ of the reconstructed image results from the superposition of M such spokes over π radians.

$$S(k_x, k_y) = \frac{\sigma^2}{L} \Delta\theta \sum_{m=0}^{M-1} \delta\{k_x \sin(m\Delta\theta) - k_y \cos(m\Delta\theta)\}$$

$$|G(k)|^2 \sum_{i=-\infty}^{\infty} \text{rect}(k-2i/a) |H_s(k-2i/a)|^2 |H_i(k-2i/a)|^2 \quad (\text{II.7})$$

where k_x and k_y are rectilinear frequency space coordinates, $\Delta\theta$ is the angular sample spacing between the projections and $\delta\{k_x \sin(m\Delta\theta) - k_y \cos(m\Delta\theta)\}$ is a 2-D delta function which lies along a spoke in frequency space. The NPS described by equation (II.7) does not consider sampling of the reconstructed image. The NPS for a sampled 2-D image can be computed from (II.7) as the following.

$$S_{\text{samp}}(k_x, k_y) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} S(k_x - i/a, k_y - j/a) \quad (\text{II.8})$$

where the pixel spacing in two Cartesian directions is the same as the sampling distance of the measured projections. Therefore, for an ideal CT system, the set of equations (II.1) to (II.8) provide the NPS in the reconstructed images.

III CONJUGATE GRADIENT METHOD (CGM)

Although, conjugate gradient method is a multidimensional optimization approach for the quadratic functional, it was originally designed to solve the system of linear equations (III.1) by minimizing the corresponding quadratic form (III.2).

$$\mathbf{A}\mathbf{X} = \mathbf{B} \quad (\text{III.1})$$

$$\mathbf{J}(\mathbf{X}) = \frac{1}{2} \mathbf{X}^T \mathbf{A} \mathbf{X} - \mathbf{B}^T \mathbf{X} \quad (\text{III.2})$$

where $\mathbf{A}$ is a real, symmetric and positive definite matrix. The equivalence of these two problems is established by taking the gradient of $\mathbf{J}$ such that

$$\Delta \mathbf{J}(\mathbf{X}) = \mathbf{0} = \mathbf{A} \mathbf{X} - \mathbf{B} \quad (\text{III.3})$$

Therefore, the vector $\mathbf{X}'$ that minimizes (III.2) is also the solution of (III.1).

Proceeding from an arbitrary search point $\mathbf{X}_0$, CGM determines a point which is successively closer to the solution as follows.

$$\mathbf{X}_{i+1} = \mathbf{X}_i + \alpha_i \mathbf{P}_i \quad (\text{III.4})$$

where α_i is a positive scalar that defines the distance between two consecutive points in the solution space, and $\mathbf{P}_i$ is the search vector. The minimum of $J(\mathbf{X})$ along $\mathbf{P}_i$ occurs at $\mathbf{X}_{i+1}$ where $\mathbf{P}_i$ is tangent to the family of contours defined by (III.2) such that

$$\Delta \mathbf{J}^T(\mathbf{X}_{i+1}) \mathbf{P}_i = \mathbf{P}_i^T \Delta \mathbf{J}(\mathbf{X}_{i+1}) = 0 \quad (\text{III.5})$$

Using iteration (III.4), the solution vector at the n^{th} iteration can be expressed in terms of the vector at the i^{th} iteration as follows

$$\mathbf{X}_n = \mathbf{X}_i + \sum_{j=1}^{n-1} \alpha_j \mathbf{P}_j \quad (\text{III.6})$$

Pre-multiplying each term of (III.6) by $\mathbf{A}$, subtracting $\mathbf{B}$ from the both sides, and using the definition of the gradient, one can write

$$\Delta \mathbf{J}(\mathbf{X}_n) = \Delta \mathbf{J}(\mathbf{X}_i) + \sum_{j=1}^{n-1} \alpha_j \mathbf{A} \mathbf{P}_j \quad (\text{III.7})$$

A criterion for defining the search direction vector $\mathbf{P}_j$ can be developed by pre-multiplying equation (III.7) by $\mathbf{P}_{i-1}$ which gives

$$(\mathbf{P}_{i-1})^T \Delta \mathbf{J}(\mathbf{X}_n) = (\mathbf{P}_{i-1})^T \Delta \mathbf{J}(\mathbf{X}_i) + \sum_{j=1}^{n-1} \alpha_j (\mathbf{P}_{i-1})^T \mathbf{A} \mathbf{P}_j \quad (\text{III.8})$$

The first term on the right-hand-side of equation (III.8) vanishes due to equation (III.5), and if we choose the search directions such that

$$(\mathbf{P}_i)^T \mathbf{A} \mathbf{P}_j = 0, \text{ for } i \neq j \quad (\text{III.9})$$

then the summation on the right-hand-side also reduces to zero, so that

$$(\mathbf{P}_{i-1})^T \Delta \mathbf{J}(\mathbf{X}_n) = 0 \quad (\text{III.10})$$

The condition expressed in equation (III.9) is known as $\mathbf{A}$ -conjugacy and is sufficient to prove that the search directions $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_{n-1}$ are linearly independent. Therefore, any vector $\Delta \mathbf{J}(\mathbf{X}_n)$ in the n -dimensional space, where n being the number of elements in the solution vector $\mathbf{X}'$, can be expressed as a linear combination of these basis search directions. The equation (III.10) will be non-zero for at least one i unless $\Delta \mathbf{J}(\mathbf{X}_n) = 0$. Therefore, the conditions that the search directions $\mathbf{P}_i$ are $\mathbf{A}$ -conjugate causes the gradient

vector $\Delta J(X_n)$ to vanish identically at the n^{th} iteration. This gives a sufficient condition that the vector X_n is the solution of the system of equation (III.1). In addition, it proves that there cannot be more than n iterations to reach the solution.

The search direction is related to the gradient direction as follows

$$P_i = -\Delta J(X_i) + \beta_i P_{i-1} \quad (\text{III.11})$$

and $P_0=0$. The scalars β_i must be determined to satisfy the A-conjugacy after writing equation (III.9) as follows.

$$(P_{i-1})^T A P_i = 0 \quad (\text{III.12})$$

Substituting P_i from equation (III.11) gives

$$-(P_{i-1})^T A \Delta J(X_i) + \beta_i (P_{i-1})^T A P_{i-1} = 0 \quad (\text{III.13})$$

Equation (III.13) can be solved for β_i as follows

$$\beta_i = \frac{(P_{i-1})^T A \Delta J(X_i)}{(P_{i-1})^T A P_{i-1}} \quad (\text{III.14})$$

The major computational effort is spent in multiplying the matrix A with any vector, thus it is advantageous to calculate β_i in such a way that matrix multiplication $A \Delta J(X_i)$ is avoided. It can be shown that the parameters β_i can alternatively be computed as follows

$$\beta_i = \frac{\Delta J^T(X_i) \Delta J(X_i)}{\Delta J^T(X_{i-1}) \Delta J(X_{i-1})} \quad (\text{III.15})$$

The parameter α in equation (III.4) is estimated to make sure that the quadratic form (III.2) is minimized along the search direction P_i , which can be easily accomplished by using (III.5) such that

$$P_i^T \Delta J(X_{i+1}) = P_i^T (A X_i + \alpha_i A P_i - B) = 0 \quad (\text{III.16})$$

or

$$\alpha_i = \frac{P_i^T \Delta J(X_i)}{P_i^T A P_i} \quad (\text{III.17})$$

The CGM method can be summarized as follows. From an initial point X_0 , the

vector close to the solution can be found using equations (III.4), (III.11), (III.15) and (III.17). In the i^{th} iteration, the gradient $\Delta J(X_i)$ is first computed which does not require multiplication AX_i if AP_{i-1} is already computed in (III.17) for the $(i-1)^{\text{th}}$ iteration since $\Delta J(X_i) = \Delta J(X_{i-1}) + \alpha_{i-1}AP_{i-1}$. Then the parameter β is computed from equation (III.15) using the previous gradient $\Delta J(X_{i-1})$. Using the previous search direction P_{i-1} , $\Delta J(X_i)$, and β , the i^{th} search direction P_i can be computed from (III.11). Finally, the value of α_i computed from (III.17) which requires multiplication AP_i , and the search direction P_i is used in equation (III.4) to proceed to the next point. This iteration is continued until the gradient vector vanishes to zero.

Although, the CGM is supposed to find the solution in no more than n iterations, due to roundoff errors the A -conjugacy property is violated after a certain number of iterations. Therefore, it is advisable to set the search direction equal to the gradient direction after a few iterations.